The Laplace Transform

Definition: The Laplace transform $\mathcal{L}\{f(t)\}$ of a function $f(t)$ defined on $[0,\infty)$ is the function $F(s)$ given by

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \lim_{R\to\infty}\int_0^R f(t)e^{-st}dt = F(s),$$

(if this limit exists). Note that s may be complex.

Linearity: The Laplace transform is a *linear* transformation (by nature of linearity of integration):

$$\begin{aligned}\mathcal{L}\{a(f(t))+b(g(t))\} &= \int_0^\infty (af(t)+bg(t))e^{-st}dt \\ &= \int_0^\infty (af(t))e^{-st}dt + \int_0^\infty (bg(t))e^{-st}dt \\ &= a\int_0^\infty f(t)e^{-st}dt + b\int_0^\infty g(t)e^{-st}dt \\ &= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}\end{aligned}$$

Inverse Laplace Transform: A function $f(t)$ whose Laplace transform is $F(s)$ is called the **inverse Laplace transform** of F, denoted $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Determine the Laplace transform of each of the following functions.

1) $f(t) = 1$

$$\mathcal{L}\{1\} = \frac{1}{s} \quad (s>0)$$

2) $f(t) = e^{at}$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad (s>a)$$

Know about "exponential shift": $= \int_0^\infty f(t) e^{-\alpha t} = F(\alpha)$, $\alpha = s-a$

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{at} f(t) e^{-st} dt = \boxed{\int_0^\infty f(t) e^{-(s-a)t} dt} = F(s-a)$$

Principle: If the Laplace transform for f(t) is F(s), the Laplace transform for the function e^{at} f(t) is simply a shift of F(s) by a units.

Ex: $\mathcal{L}\{t^2\} = \dfrac{2}{s^3}$

So, $\mathcal{L}\{e^{4t} t^2\} = \dfrac{2}{(s-4)^3}$

Ex: Use exponential shift & linearity:

$$\mathcal{L}\{e^{3t}(t^2 + 2t + 5)\}$$

Linearity:

$$= \mathcal{L}\{e^{3t} t^2\} + 2\mathcal{L}\{e^{3t} \cdot t\} + 5\mathcal{L}\{e^{3t}\}$$

Exponential Shift:

$$= \frac{2}{(s-3)^3} + \frac{2 \cdot 1}{(s-3)^2} + \frac{5}{s-3}.$$

Determine the Laplace transform of each of the following functions.

1) $f(t) = \cosh(bt) = (e^{bt} + e^{-bt})/2$ (Use linearity and exponential shift.)

$$\mathcal{L}\{\cosh(bt)\} = \frac{1}{2}\left[\mathcal{L}\{e^{bt}\} + \mathcal{L}\{e^{-bt}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-b} + \frac{1}{s+b}\right]$$

$$= \frac{s+b+s-b}{2(s^2-b^2)} = \boxed{\frac{s}{s^2-b^2}}$$

2) $f(t) = \sinh(bt) = (e^{bt} - e^{-bt})/2$ (Use linearity and exponential shift.)

$$\mathcal{L}\{\sinh(bt)\} = \frac{1}{2}\left[\mathcal{L}\{e^{bt}\} - \mathcal{L}\{e^{-bt}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-b} - \frac{1}{s+b}\right]$$

$$= \frac{1}{2}\left[\frac{s+b-(s-b)}{s^2-b^2}\right]$$

$$\boxed{= \frac{b}{s^2-b^2}}$$

Recall that

$$e^{(bt)i} = \cos(bt) + i\sin(bt) \qquad (1)$$

and

$$e^{(-bt)i} = \cos(-bt) + i\sin(-bt)$$

$$= \cos(bt) - i\sin(bt) \qquad (2)$$

Add expressions (1) and (2) and solve for $\boxed{\cos(t) = (e^{(bt)i}+e^{(-bt)i})/2.}$

Subtract expression (2) from (1) and solve for $\boxed{\sin(t) = (e^{(bt)i}-e^{(-bt)i})/(2i).}$

Use the results from the previous slide to determine the Laplace transforms of sin(bt) and cos(bt). Use linearity and exponential shift.

$$\cos(bt) = \frac{e^{ibt} + e^{-ibt}}{2}$$

$$\mathcal{L}\{\cos(bt)\} = \frac{1}{2}\left[\frac{1}{s-bi} + \frac{1}{s+bi}\right]$$

$$= \frac{1}{2}\left[\frac{s+bi+s-bi}{s^2+b^2}\right]$$

$$= \frac{s}{s^2+b^2}$$

$$\mathcal{L}\{\sin(bt)\} = \frac{1}{2i}\left[e^{ibt} - e^{-ibt}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-bi} - \frac{1}{s+bi}\right]$$

$$= \frac{1}{2i}\left[\frac{s+bi-(s-bi)}{s^2+b^2}\right]$$

$$= \boxed{\frac{b}{s^2+b^2}}$$