DECOUPLING LINEAR SYSTEMS:

$$x' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} x \longrightarrow w' = \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix} w.$$

$$\begin{cases} x_1' = ax_1 + bx_2 \\ x_2' = cx_1 + dx_2 \end{cases} \longrightarrow \begin{cases} w_1' = hw_1 \\ w_2' = kw_2 \end{cases}$$

Given $\vec{x}\,' = A\vec{x}$, Let $\vec{x} = P\vec{w}$

where $A = PDP^{-1}$. $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$P = [\vec{v}_1 \ \vec{v}_2]$

With $\underline{\vec{x} = P\vec{w}}$, $\vec{w} = P^{-1}\vec{x}$.

$$\begin{aligned} \vec{w}\,' &= P^{-1}\vec{x}\,' \\ &= P^{-1}(A\vec{x}) \\ &= P^{-1}(PDP^{-1}\vec{x}) \\ &= DP^{-1}\vec{x} \\ &= D\vec{w} \end{aligned}$$

$$\vec{w}\,' = D\vec{w}$$

$$\boxed{\begin{aligned} w_1' &= \lambda_1 w_1 \\ w_2' &= \lambda_2 w_2 \end{aligned}}$$

easy to solve!

$$\begin{cases} w_1 = c_1 e^{\lambda_1 t} \\ w_2 = c_2 e^{\lambda_2 t} \end{cases}$$

Then $\vec{x} = P\vec{w} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}.$

$$\vec{x}' = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \vec{x} \qquad \vec{x} = P\vec{w}.$$

$$\vdots$$

$$\vec{w}' = D\vec{w}$$

$$\begin{cases} x_1' = 2x_1 + 2x_2 \\ x_2' = x_1 + 3x_2 \end{cases}$$

$$\vec{w}' = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \vec{w}$$

$\lambda_1 = 4, \quad \lambda_2 = 1$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$w_1' = 4w_1$

$w_2' = w_2$

$\Rightarrow$ $w_1 = c_1 e^{4t}$, $w_2 = c_2 e^{t}$

$$\vec{x} = P\vec{w}$$

$$\vec{x} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 e^{4t} \\ c_2 e^{t} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} (1) c_1 e^{4t} + (2) c_2 e^{t} \\ (1) c_1 e^{4t} + (-1) c_2 e^{t} \end{bmatrix}$$

solution vector;

components of solution:

$x_1 = c_1 e^{4t} + 2c_2 e^{t}$

$x_2 = c_1 e^{4t} - c_2 e^{t}$

If we have IVP,

$\vec{x} = P\vec{w}$

$\vec{x}_0 = P\vec{c}$

$\vec{c} = P^{-1}\vec{x}_0$

If $\vec{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

$P = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} \cdot -\frac{1}{3}$

$= \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix}$

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4/3 \\ 1/3 \end{bmatrix}$$

So $$\vec{x}(t) = \begin{bmatrix} \frac{4}{3} e^{4t} + \frac{2}{3} e^{t} \\ \frac{4}{3} e^{4t} - \frac{1}{3} e^{t} \end{bmatrix}$$

$$\begin{aligned} x_1' &= -3x_1 + x_2 \\ x_2' &= x_1 - 3x_2 + e^{-t} \end{aligned}$$

$$\vec{x}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

$$\vec{x}' = A\vec{x} + \vec{f}(t)$$

If $A = PDP^{-1}$ and $\vec{x} = P\vec{w}$:

$$\vec{w} = P^{-1}\vec{x}$$

$$\vec{w}' = P^{-1}\vec{x}'$$

$$\vec{w}' = P^{-1}(A\vec{x} + \vec{f})$$

$$\vec{w}' = P^{-1}A\vec{x} + P^{-1}\vec{f}$$

$$\vec{w}' = P^{-1}PDP^{-1}\vec{x} + P^{-1}\vec{f}$$

$$\vec{w}' = DP^{-1}\vec{x} + P^{-1}\vec{f}$$

$$\boxed{\vec{w}' = D\vec{w} + P^{-1}\vec{f}.}$$

$$x_1' = -3x_1 + x_2$$
$$x_2' = x_1 - 3x_2 + e^{-t}$$

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$

$$\lambda^2 + 6\lambda + 8 = 0$$

$$\boxed{\lambda_1 = -4, \quad \lambda_2 = -2}$$

$$(A - \lambda_1 I)\vec{v}_1 = \vec{0} \implies \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\vec{v}_1 = \vec{0} \implies \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$(A - \lambda_2 I)\vec{v}_2 = \vec{0} \implies \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\vec{v}_2 = \vec{0} \implies \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So $D = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$\vec{w}' = D\vec{w} + P^{-1}\vec{f}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\vec{w}' = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix}\vec{w} + \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

$$w_1' = -4w_1 - \tfrac{1}{2}e^{-t} \qquad w_2' = -2w_2 + \tfrac{1}{2}e^{-t}$$

Int. factor: (1st order linear)

$$w_1' + 4w_1 = -\tfrac{1}{2}e^{-t}$$

$$p(t) = e^{\int 4\,dt} = e^{4t}$$

$$w_1'e^{4t} + 4e^{4t}w_1 = -\tfrac{1}{2}e^{-t}e^{4t}$$

$$\frac{d}{dt}\left(w_1 e^{4t}\right) = -\tfrac{1}{2}e^{3t}$$

$$w_1 e^{4t} = -\tfrac{1}{6}e^{3t} + C_1$$

$$\boxed{w_1 = -\tfrac{1}{6}e^{-t} + C_1 e^{-4t}}$$

Undetermined coefficients:

$$w_h = C_2 e^{-2t}$$

$$w_p = Ae^{-t}$$

$$w_p' = -Ae^{-t}$$

So eqn. becomes

$$-Ae^{-t} = -2Ae^{-t} + \tfrac{1}{2}e^{-t}$$

$$A = \tfrac{1}{2} \implies w_p = \tfrac{1}{2}e^{-t}$$

So $\boxed{w_2 = C_2 e^{-2t} + \tfrac{1}{2}e^{-t}}$

$$\vec{x} = P\vec{w} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{6}e^{-t} + C_1 e^{-4t} \\ \frac{1}{2}e^{-t} + C_2 e^{-2t} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} \left(-\frac{1}{6}+\frac{1}{2}\right)e^{-t} + C_1 e^{-4t} + C_2 e^{-2t} \\ \left(\frac{1}{6}+\frac{1}{2}\right)e^{-t} - C_1 e^{-4t} + C_2 e^{-2t} \end{bmatrix}$$

$$b(n+1) = b(n) \cdot \frac{1}{2} + 10$$

$$b(n+1) - b(n) = \frac{b(n)}{2} - b(n) + 10$$

$$b(n+1) - b(n) = -\frac{b(n)}{2} + 10$$

$$b(n+\Delta n) - b(n) = -\Delta n \cdot \frac{1}{2} \cdot b(n) + \Delta n \cdot 10$$

$$\lim_{\Delta n \to 0} \frac{b(n+\Delta n) - b(n)}{\Delta n} = \lim_{\Delta n \to 0} -\frac{1}{2} b(n) + 10$$

$$\boxed{b'(n) = -\frac{1}{2} b(n) + 10}$$

$$b'(t) = -\frac{1}{2} b(t) + 10$$

$$b_h = c_1 e^{-\frac{1}{2}t} \qquad b_p = A$$

$$0 = -\frac{1}{2} A + 10 \qquad \boxed{A = 20}$$

$$\boxed{b(t) = c_1 e^{-\frac{1}{2}t} + 20}$$

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Given $\vec{x}' = A\vec{x}$, Let $\vec{x} = P\vec{w}$

where $A = PDP^{-1}$. $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$P = [\vec{v}_1 \ \vec{v}_2]$

With $\underline{\vec{x} = P\vec{w}}$, $\vec{w} = P^{-1}\vec{x}$.

$$\begin{aligned} \vec{w}' &= P^{-1}\vec{x}' \\ &= P^{-1}(A\vec{x}) \\ &= P^{-1}(PDP^{-1}\vec{x}) \\ &= DP^{-1}\vec{x} \\ &= D\vec{w} \end{aligned}$$

$$\vec{w}' = D\vec{w}$$

$$\boxed{\begin{aligned} w_1' &= \lambda_1 w_1 \\ w_2' &= \lambda_2 w_2 \end{aligned}}$$

easy to solve!

$$\begin{cases} w_1 = c_1 e^{\lambda_1 t} \\ w_2 = c_2 e^{\lambda_2 t} \end{cases}$$

Then $\vec{x} = P\vec{w} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}$.

With $\vec{x} = P\vec{w}$, $\vec{x}\,' = A\vec{x}$, $A = PDP^{-1}$,

we have:

$$\vec{w} = P^{-1}\vec{x}.$$

$$\vec{w}\,' = P^{-1}\vec{x}\,'$$

$$= P^{-1}(A\vec{x})$$

$$= P^{-1}(PDP^{-1}\vec{x})$$

$$= DP^{-1}\vec{x}$$

$$\boxed{\vec{w}\,' = D\vec{w}.}$$

Start here. :)

$$\vec{w}\,' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \vec{w}$$

$$\begin{cases} w_1' = \lambda_1 w_1 \\ w_2' = \lambda_2 w_2 \end{cases}$$

$$\Longrightarrow \begin{cases} w_1 = c_1 e^{\lambda_1 t} \\ w_2 = c_2 e^{\lambda_2 t} \end{cases}$$

Then $\vec{x} = P\vec{w}$.

Let $\vec{x}' = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \vec{x}$

$$\begin{cases} \lambda_1 = 4, & \lambda_2 = 1 \\ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \end{cases}$$

<u>Solution:</u> $X(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$

<u>Solution via decoupling:</u>

$\vec{w}' = D\vec{w} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \vec{w}$ $\quad w_1' = 4w_1$

$w_2' = w_2$

$$\Longrightarrow \boxed{\begin{aligned} w_1 &= c_1 e^{4t} \\ w_2 &= c_2 e^{t} \end{aligned}}$$

$\vec{x} = P\vec{w} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} =$

IVP ... suppose $\vec{X}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$

$\vec{x} = P\vec{w}$; but if $t=0$

$\vec{X}_0 = P\vec{c}$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{2}{3} \end{bmatrix}$$

So $x(t) = \begin{bmatrix} c_1 e^{4t} + 2c_2 e^{t} \\ c_1 e^{4t} - c_2 e^{t} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} e^{4t} + \frac{4}{3} e^{t} \\ \frac{5}{3} e^{4t} - \frac{2}{3} e^{t} \end{bmatrix}.$

$$\begin{aligned} x_1' &= -3x_1 + x_2 \\ x_2' &= x_1 - 3x_2 + e^{-t} \end{aligned}$$

in which

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}.$$

The eigenvalues and corresponding eigenvectors of A are

$$\lambda_1 = -2 \quad \text{and} \quad \lambda_2 = -4 \quad \text{with} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\vec{x}' = A\vec{x} + \vec{f}(t)$;

$$\vec{x}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

Let $\vec{x} = P\vec{w}$. So $\vec{w} = P^{-1}\vec{x}$.

$$\begin{aligned} \vec{w}' &= P^{-1}\vec{x}' \\ &= P^{-1}\left(A\vec{x} + \vec{f}(t)\right) \\ &= P^{-1}A\vec{x} + P^{-1}\vec{f} \\ &= P^{-1}(PDP^{-1}\vec{x}) + P^{-1}\vec{f} \\ &= DP^{-1}\vec{x} + P^{-1}\vec{f} \end{aligned}$$

$$\boxed{\vec{w}' = D\vec{w} + P^{-1}\vec{f}.}$$

$$\vec{x}' = A\vec{x} + \vec{f}(t);$$

$$\vec{x}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

$$\vec{w}' = D\vec{w} + P^{-1}\vec{f}.$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad D = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} +1/2 & +1/2 \\ +1/2 & -1/2 \end{bmatrix}$$

$$\vec{w}' = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}\vec{w} + \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

$$w_1' = -2w_1 + \tfrac{1}{2}e^{-t} \qquad w_2' = -4w_2 - \tfrac{1}{2}e^{-t}$$

(undetermined coefficients)

$w_h = c_1 e^{-2t}$

Let $w_p = Ae^{-t}$

The eqn. becomes

$-Ae^{-t} = -2Ae^{-t} + \tfrac{1}{2}e^{-t}$

$A = 1/2,$

So $w_1 = w_h + w_p$

$$\boxed{w_1 = c_1 e^{-2t} + \tfrac{1}{2}e^{-t}}$$

(integrating factor)

$w_2' + 4w_2 = -\tfrac{1}{2}e^{-t}$

$p(t) = e^{\int 4\,dt} = e^{4t}$

mult through by e^{4t}:

$w_2' e^{4t} + 4e^{4t}w_2 = -\tfrac{1}{2}e^{-t}e^{4t}$

$\frac{d}{dt}\left(w_2 e^{4t}\right) = -\tfrac{1}{2}e^{3t}$

$w_2 e^{4t} = -\tfrac{1}{6}e^{3t} + C_2$

$$\boxed{w_2 = -\tfrac{1}{6}e^{-t} + C_2 e^{-4t}}$$

$$\vec{w} = \begin{bmatrix} c_1 e^{-2t} + \tfrac{1}{2}e^{-t} \\ c_2 e^{-4t} - \tfrac{1}{6}e^{-t} \end{bmatrix}$$

$$\vec{x} = P\vec{w} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} =$$

$$\vec{x}(t) = \begin{bmatrix} c_1 e^{-2t} + c_2 e^{-4t} + \left(\tfrac{1}{2} - \tfrac{1}{6}\right)e^{-t} \\ c_1 e^{-2t} - c_2 e^{-4t} + \left(\tfrac{1}{2} + \tfrac{1}{6}\right)e^{-t} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Lab Recap:

$$b(n+1) = \frac{1}{2}b(n) + 10.$$

DISCRETE: $\Delta b(n) = b(n+1) - b(n)$

$$\boxed{\Delta b(n) = -\frac{1}{2}b(n) + 10.}$$

CONTINUOUS:

$$b(n+1) - b(n) = -\frac{1}{2}b(n) + 10$$

$$b(n+\Delta n) - b(n) = -\frac{1}{2}\Delta n\, b(n) + \Delta n(10)$$

$$\frac{b(n+\Delta n) - b(n)}{\Delta n} \doteq -\frac{1}{2}b(n) + 10$$

$$\lim_{\Delta n \to 0} \frac{b(n+\Delta n) - b(n)}{\Delta n} = \lim_{\Delta n \to 0}\left(-\frac{1}{2}b(n) + 10\right)$$

$$b'(n) = -\frac{1}{2}b(n) + 10$$

$$b'(n) + \frac{1}{2}b(n) = 10$$

int. factor: $p(n) = e^{\int \frac{1}{2} dn} = e^{n/2}.$

$$b' \cdot e^{n/2} + \frac{1}{2}e^{n/2}b = 10e^{n/2}$$

$$\frac{d}{dn}\left(e^{n/2}b\right) = 10e^{n/2}$$

$$e^{n/2}b(n) = 20e^{n/2} + C$$

$$\boxed{b(n) = 20 + Ce^{-n/2}}$$