LAPLACE TRANSFORMS

The Laplace Transform $\mathcal{L}\{f(t)\}$ of a "suitable" function $f(t)$ defined on $[0,\infty)$ is the function $F(s)$ given by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \lim_{b\to\infty} \int_0^b e^{-st} f(t)\,dt$$

(where s <u>may</u> be complex!)

<u>KEY</u>: Laplace transforms are linear, i.e., $\mathcal{L}\{af(t)+bg(t)\}$

$$= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

(defined for all s for which each transform is defined).

Find $\mathcal{L}\{1\}$. (i.e., $f(t)=1$)

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \lim_{b\to\infty}\left[\cancel{\frac{e^{-sb}}{-s}} - \frac{e^{-s(0)}}{-s}\right] = \frac{1}{s}$$

0

$s>0$ $(s>0)$

$$\mathcal{L}\{1\} = \frac{1}{s} \iff \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1.$$

for $s>0$

The Laplace Transform of $f(t)=1$ is $F(s)=\frac{1}{s}$ for $s>0$.

The inverse Laplace Transform of $F(s)=\frac{1}{s}$ is $f(t)=1$.

<u>Ex</u>: $\mathcal{L}\{-3\} = -3\cdot\frac{1}{s} = \frac{-3}{s}$

(by linearity of $\mathcal{L}$)

Find $\mathcal{L}\{e^{at}\}$.

$$F(s)=\int_0^\infty e^{at}e^{-st}\,dt=\int_0^\infty e^{(a-s)t}\,dt$$

$$=\left[\frac{e^{(a-s)t}}{(a-s)}\right]_0^\infty$$

$$=\lim_{b\to\infty}\left[\underbrace{\cancel{\frac{e^{(a-s)b}}{a-s}}}_{0\text{ if } s>a}-\frac{e^{(a-s)(0)}}{a-s}\right]=\boxed{\frac{1}{s-a},\ s>a}$$

$$\mathcal{L}\{e^{at}\}=\frac{1}{s-a}\ (s>a)\iff\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}=e^{at}$$

<u>Ex</u>: $\mathcal{L}\{5e^{-2t}\}=5\mathcal{L}\{e^{-2t}\}$

(by linearity of $\mathcal{L}$)

$$=5\cdot\frac{1}{s-(-2)}$$

$$=\frac{5}{s+2}.$$

$$\mathcal{L}\{t\}=\int_0^\infty te^{-st}\,dt$$

$\int u\,dv=uv-\int v\,du$

$u=t \quad dv=e^{-st}dt$

$du=dt \quad v=\dfrac{e^{-st}}{-s}$

$$=\left[\frac{te^{-st}}{-s}+\int\frac{e^{-st}}{s}\,dt\right]_0^\infty$$

$$=\left[\frac{te^{-st}}{-s}+\frac{e^{-st}}{-s^2}\right]_0^\infty$$

$$=\lim_{b\to\infty}\left[\left(\underbrace{\cancel{\frac{be^{-sb}}{-s}}}_{0 \text{ if } s>0}+\underbrace{\cancel{\frac{e^{-sb}}{-s^2}}}_{0}\right)-\left(\frac{0}{-s}+\frac{1}{-s^2}\right)\right]=\frac{1}{s^2}\ (s>0)$$

$$\mathcal{L}\{t\}=\frac{1}{s^2}\ (s>0).\iff\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}=t.$$

$$\mathcal{L}\{t\} = \int_0^\infty t e^{-st}\,dt$$

$u = t \qquad dv = e^{-st}dt$

$du = dt \qquad v = \frac{e^{-st}}{-s}$

$$= \left[\frac{te^{-st}}{-s} - \int \frac{e^{-st}}{-s}\,dt\right]_0^\infty$$

$$= \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\,dt\right]_0^\infty$$

$$= \lim_{b\to\infty}\left[\left(\cancel{\frac{be^{-sb}}{-s}}_{\to 0} - \cancel{\frac{e^{-sb}}{s^2}}_{\to 0}\right) - \left(\frac{0e^{-0}}{-s} - \frac{e^{-0}}{s^2}\right)\right]$$

(if $s>0$)

$$= \frac{1}{s^2}$$

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2},\ s>0}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$$

In general,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$ (if n is an integer).

$$= \frac{\Gamma(n+1)}{s^{n+1}}$$ If n is not an integer.

Ex: $\mathcal{L}\{t^5\} = \frac{5!}{s^6} = \frac{120}{s^6}.$

Using Linearity:

$$\mathcal{L}\{4e^{2t}+3e^{-3t}-2t^2+4t+6\}$$

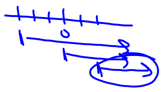
We could:

$$\int_0^\infty (4e^{2t}+3e^{-3t}-2t^2+4t+6)e^{-st}\,dt = \ldots$$

OR:

$$F(s)=\frac{4}{s-2}+\frac{3}{s+3}-\frac{2\cdot 2!}{s^3}+\frac{4}{s^2}+\frac{6}{s}$$

$s>2 \quad s>-3 \quad s>0$



Use Linearity:

$$\mathcal{L}\{e^{3t}+t^2-4t+2\}$$

$$=\mathcal{L}\{e^{3t}\}+\mathcal{L}\{t^2\}-4\mathcal{L}\{t\}+2\mathcal{L}\{1\}.$$

$$=\frac{1}{s-3}+\frac{2}{s^3}-\frac{4}{s^2}+\frac{2}{s} \quad (s>3)$$

$$=\frac{s^3+2(s-3)-4s(s-3)+2s^2(s-3)}{s^3(s-3)}$$

$$=\frac{3s^3-10s^2+14s-6}{s^3(s-3)}=\mathcal{L}\{e^{3t}+t^2-4t+2\}.$$

hyperbolic:

$$\cosh(kx)=\frac{e^{kx}+e^{-kx}}{2}$$

$$\sinh(kx)=\frac{e^{kx}-e^{-kx}}{2}$$

$$\mathcal{L}\{\cosh(kx)\}=\frac{1}{2}\left(\frac{1}{s-k}+\frac{1}{s+k}\right)$$

$$=\frac{1}{2}\left(\frac{s+k+s-k}{s^2-k^2}\right)$$

$$=\boxed{\frac{s}{s^2-k^2}}$$

$$\mathcal{L}\{\sinh(kx)\}=\frac{1}{2}\left(\frac{1}{s-k}-\frac{1}{s+k}\right)$$

$$=\frac{1}{2}\left(\frac{s+k-(s-k)}{s^2-k^2}\right)$$

$$=\boxed{\frac{k}{s^2-k^2}}$$

$$\mathcal{L}\{\cos(kx)\} = \int_0^\infty \cos(kx)e^{-sx}\,dx$$

(leads to "wraparound" int. by parts...)

Instead, we'll "complexify":

$$e^{ikx} = \cos(kx) + i\sin(kx). \qquad (1)$$

$$e^{-ikx} = \cos(-kx) + i\sin(-kx)$$

$$= \cos(kx) - i\sin(kx). \qquad (2)$$

$$\cos(kx) = \frac{e^{ikx} + e^{-ikx}}{2}$$

$$\hookrightarrow \mathcal{L}\{\cos(kx)\} = \frac{1}{2}\left(\frac{1}{s-ik} + \frac{1}{s+ik}\right)$$

$$= \frac{1}{2}\left(\frac{s+ik+s-ik}{s^2+k^2}\right)$$

$$= \boxed{\frac{s}{s^2+k^2}}$$

$$\sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i}$$

$$\mathcal{L}\{\sin(kx)\} = \frac{1}{2i}\left(\frac{1}{s-ki} - \frac{1}{s+ki}\right)$$

$$= \frac{1}{2i}\left(\frac{s+ki-(s-ki)}{s^2+k^2}\right)$$

$$= \boxed{\frac{k}{s^2+k^2}}$$

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$	
1	$1/s$	$s>0$
t	$1/s^2$	$s>0$
t^n, n an integer	$n!/s^{n+1}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$s>0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$s>0$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$	$s>\lvert k\rvert$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$	$s>\lvert k\rvert$

Backwards?

$$F(s)=\frac{2s-14}{(s+1)(s-3)}=\frac{A}{s+1}+\frac{B}{s-3}$$

$$=\frac{4}{s+1}-\frac{2}{s-3}$$

$$=4\cdot\frac{1}{s+1}-2\cdot\frac{1}{s-3}$$

$$=\mathcal{L}\left\{4e^{-t}-2e^{3t}\right\}.$$

$$F(s)=\frac{2s^2-s+11}{\boxed{(s-1)(s^2+5)}}=\frac{A}{s-1}+\frac{Bs+C}{s^2+5}$$

$$2s^2-s+11=A(s^2+5)+(Bs+C)(s-1)$$

$$2s^2-s+11=As^2+5A+Bs^2-Bs+Cs-C$$

$\underline{\underline{s^2}}$: $2=A+B$ $\qquad A=2,\ B=0,\ C=-1.$

$\underline{\underline{s}}$: $-1=-B+C$ $\Rightarrow F(s)=\dfrac{2}{s-1}-\dfrac{1}{s^2+5}$

$\underline{\underline{1}}$: $11=5A-C$

$$F(s)=\frac{2}{s-1}-\frac{1}{s^2+5}$$

$$=2\cdot\frac{1}{s-1}-\frac{1}{\sqrt{5}}\frac{\sqrt{5}}{s^2+(\sqrt{5})^2}$$

$$=\mathcal{L}\left\{2e^{t}-\frac{1}{\sqrt{5}}\sin(\sqrt{5}t)\right\}.$$

$$F(s) = \frac{s+1}{s^2+4s+13} = \frac{s+1}{(s+2)^2+9}$$

$$(s+2)^2+9$$

$$= \frac{s+2-1}{(s+2)^2+9}$$

$$e^{at}\cos kt = \frac{s-a}{(s-a)^2+k^2} \qquad = \frac{s+2}{(s+2)^2+9} - \frac{1}{(s+2)^2+9}$$

$$e^{at}\sin kt = \frac{k}{(s-a)^2+k^2} \qquad = \mathcal{L}\left\{e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t\right\}$$