"DECOUPLING" a system means to

Convert $\vec{x}' = A\vec{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

into $\vec{w}' = D\vec{w} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

Assume $\vec{x}' = A\vec{x}$, i.e., that

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

and that A is a diagonalizable matrix.

Then, $AP = PD \implies A = PDP^{-1}$

eigenvectors $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$

eigenvalues (diagonal) $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

To decouple a system $\vec{x}' = A\vec{x}$,

Let $\vec{x} = P\vec{w}$.

eigenvectors of $\vec{x}$

Then $\vec{w} = P^{-1}\vec{x}$

$\vec{w}' = P^{-1}\vec{x}'$

b/c $\vec{x}' = A\vec{x}$

$= P^{-1}(A\vec{x})$

b/c $A = PDP^{-1}$

$= P^{-1}(PDP^{-1}\vec{x})$

$= (P^{-1}P)DP^{-1}\vec{x}$

$\vec{w}' = DP^{-1}\vec{x}$

$P^{-1}\vec{x} = \vec{w}$.

$$\boxed{\vec{w}' = D\vec{w}}$$

So... if $\vec{x} = P\vec{w}$, then $\underbrace{\vec{w}' = D\vec{w}}_{\text{decoupled system!}}$.

Example:

$$\begin{cases} x'(t) = 2x + 2y \\ y'(t) = x + 3y \end{cases}$$

$$\vec{x}' = A\vec{x} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)(3-\lambda) - 2 = 0$$

$$\Rightarrow \quad \lambda^2 - 5\lambda + 4 = 0$$

$$\Rightarrow \quad (\lambda-4)(\lambda-1) = 0$$

$$\Rightarrow \quad \lambda_1 = 4, \quad \lambda_2 = 1.$$

For $\lambda_1 = 4$:

$$(A-\lambda_1 I)\vec{v}_1 = \vec{0} \Rightarrow \begin{bmatrix} 2-4 & 2 \\ 1 & 3-4 \end{bmatrix}\vec{v}_1 = \vec{0}$$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}\vec{v}_1 = \vec{0}$$

$$\boxed{\lambda_1 = 4 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.} \qquad (*)$$

For $\lambda_2 = 1$:

$$(A-\lambda_2 I)\vec{v}_2 = \vec{0} \Rightarrow \begin{bmatrix} 2-1 & 2 \\ 1 & 3-1 \end{bmatrix}\vec{v}_2 = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\vec{v}_2 = \vec{0}$$

$$\boxed{\lambda_2 = 1 \Rightarrow \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}} \qquad (**)$$

So $A = PDP^{-1}$

$$= \underbrace{\begin{array}{cc} v_1 & v_2 \\ \end{array}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} \overset{\lambda_1}{4} & 0 \\ 0 & \underset{\lambda_2}{1} \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix}}_{P^{-1}}$$

START

- Let $\vec{x} = P\vec{w}$, then

$$\vec{w}\,' = D\vec{w}.$$

$$\begin{bmatrix} w_1'(t) \\ w_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

$$w_1'(t) = 4w_1(t) \implies w_1(t) = c_1 e^{4t}.$$

$$w_2'(t) = w_2(t) \implies w_2(t) = c_2 e^{t}.$$

But, $\vec{x} = P\vec{w} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 e^{4t} \\ c_2 e^{t} \end{bmatrix}$

$$\boxed{\vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{4t} + 2c_2 e^{t} \\ c_1 e^{4t} - c_2 e^{t} \end{bmatrix}}$$

Recall from (*) & (**), our soln would have been

$$\vec{x(t)} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

$$= c_1 e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Given initial conditions $\vec{x}_0 = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \ldots$

$$\vec{x}_0 = P\vec{w}_0 = P\vec{c}$$

$$\Longrightarrow \vec{c} = P^{-1}(x_0)$$

$$\vec{w} = \begin{bmatrix} c_1 e^{4t} \\ c_2 e^{t} \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -2/3 \end{bmatrix}.$$

$$\boxed{\vec{X} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{4t} + 2c_2 e^{t} \\ c_1 e^{4t} - c_2 e^{t} \end{bmatrix}}$$

For $\vec{x}_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 4/3 e^{4t} - 4/3 e^{t} \\ 4/3 e^{4t} + 2/3 e^{t} \end{bmatrix}$

Decoupling:

(1) Generalizes to larger systems

(2) helps in solving nonhomogeneous.

WW #15 :)

If $\vec{x}' = A\vec{x} + \vec{f(t)}$

Let $\vec{x} = P\vec{w}$

So $\vec{w} = P^{-1}\vec{x}$.

$$\vec{w}' = P^{-1}\vec{x}'$$

$$= P^{-1}\left(A\vec{x} + \vec{f(t)}\right)$$

$$= P^{-1}A\vec{x} + P^{-1}\vec{f(t)}$$

$$= P^{-1}\underbrace{(PDP^{-1})}_{A}\vec{x} + P^{-1}\vec{f(t)}$$

$$= D\underbrace{P^{-1}\vec{x}}_{\vec{w}} + P^{-1}\vec{f(t)}$$

$$\vec{w}' = D\vec{w} + P^{-1}\vec{f(t)}$$

(still nonhomogeneous, but each component is a nonhomogeneous 1st-order eqn (from ch. 1).)

Nonhomogeneous

$$x_1'(t) = -3x_1 + x_2$$

$$x_2'(t) = x_1 - 3x_2 + e^{-t}$$

$$\vec{x}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

$$\vec{x}' = A \vec{x} + \vec{f}$$

A has $\lambda_1 = -4$ with $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\lambda_2 = -2$ with $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Start

If $\vec{x} = P\vec{w}$, then

$$\vec{w}' = D\vec{w} + P^{-1}f \quad \text{(nonhomogeneous)}$$

$$\text{So } \vec{w}' = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

$$\begin{cases} w_1'(t) = -4w_1(t) - \frac{1}{2}e^{-t} \\ w_2'(t) = -2w_2(t) + \frac{1}{2}e^{-t} \end{cases}$$

(Each of these is 1st-order nonhomogeneous linear...) $y' + py = g(t)$

$$w_1'(t) = -4w_1(t) - \tfrac{1}{2}e^{-t}$$

$$w_1' + 4w_1 = -\tfrac{1}{2}e^{-t} \qquad \text{Int factor: } e^{\int 4dt} = e^{4t}$$

$$w_1'e^{4t} + 4e^{4t}w_1 = \left(-\tfrac{1}{2}e^{-t}\right)e^{4t}$$

$$\left[w_1 \cdot e^{4t}\right]' = -\tfrac{1}{2}e^{3t}$$

$$w_1e^{4t} = -\tfrac{1}{6}e^{3t} + C_1$$

$$\boxed{w_1 = -\tfrac{1}{6}e^{-t} + c_1e^{-4t}}$$

$$\boxed{w_2'(t) = -2w_2(t)} + \tfrac{1}{2}e^{-t}$$

$$w_2'(t) + 2w_2(t) = \tfrac{1}{2}e^{-t}$$

$$w_h = c_2e^{-2t}$$

Let $w_p = Ae^{-t}$

$$w_p' = -Ae^{-t}$$

$$w_p' + 2w_p = \tfrac{1}{2}e^{-t}$$

$$-Ae^{-t} + 2Ae^{-t} = \tfrac{1}{2}e^{-t}$$

$$A = \tfrac{1}{2}e^{-t}.$$

$$\text{So } w_2(t) = w_h + w_p = \boxed{C_2e^{-2t} + \tfrac{1}{2}e^{-t}}$$

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{6}e^{-t} + c_1e^{-4t} \\ \frac{1}{2}e^{-t} + c_2e^{-2t} \end{bmatrix}$$

$$\vec{x} = P\vec{w} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{6}e^{-t} + c_1e^{-4t} \\ \frac{1}{2}e^{-t} + c_2e^{-2t} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}e^{-t} + c_1e^{-4t} + c_2e^{-2t} \\ \frac{2}{3}e^{-t} - c_1e^{-4t} + c_2e^{-2t} \end{bmatrix}$$