Repeated Eigenvalues but missing Eigenvectors ...

What if ...

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \vec{y} \quad \ldots$$

$$\begin{bmatrix} -2-\lambda & 1 \\ 0 & -2-\lambda \end{bmatrix} = \lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda+2)^2 = 0$$

$$\boxed{\lambda_{1,2} = -2}$$

algebraic multiplicity is 2

$$(A - \lambda I)\vec{v} = \vec{0}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{v} = \vec{0} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

geometric multiplicity is only 1

We say $\lambda = -2$ is a "defective" eigenvalue.

We still have at least one soln:

$$\vec{y}_1 = e^{-2t}\vec{v} = e^{-2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

But because this system is 2-dim, we need a 2nd solution independent of $\vec{y}_1$.

In the past, we'd try

$$\vec{Y}_1 = e^{\lambda t}\vec{v} \quad \text{and} \quad \vec{Y}_2 = te^{\lambda t}\vec{v}.$$

If Y_2 is a solution, $\vec{Y}_2'$ should equal $A\vec{Y}_2$.

$$\vec{Y}_2' = e^{\lambda t}\vec{v} + \lambda t e^{\lambda t}\vec{v}.$$

$$A\vec{Y}_2 = Ate^{\lambda t}\vec{v} = \lambda t e^{\lambda t}\vec{v}. \qquad A\vec{v} = \lambda\vec{v}$$

Bummer. $A\vec{Y}_2 \neq Y_2'$, i.e., $te^{\lambda t}\vec{v}$ is <u>not</u> even a solution!

Luckily, we can solve the system w/o eigenstuff:

$$\vec{Y}' = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \vec{Y}$$

$$\begin{cases} x'(t) = -2x + y \\ y'(t) = \quad -2y \end{cases}$$ — has soln $\boxed{y(t) = y_0 e^{-2t}}$

Subs. y into the first eqn:

$$x'(t) = -2x + y_0 e^{-2t} \qquad x'(t) + 2x = y_0 e^{-2t}$$

$$\boxed{x_h = x_0 e^{-2t}}$$

$$x_p = A t e^{-2t} \quad (\text{to avoid dupl. } x_h).$$

Solve for A:

$$x_p' + 2x_p = \left(A e^{-2t} - \cancel{2At} e^{-2t}\right) + \cancel{2At} e^{-2t} = y_0 e^{-2t}$$

$$\Rightarrow A = y_0.$$

$$x = x_h + x_p = \boxed{x_0 e^{-2t} + y_0 t e^{-2t}}.$$

$$\vec{Y} = \begin{bmatrix} x_0 e^{-2t} + y_0 t e^{-2t} \\ y_0 e^{-2t} \end{bmatrix} = \underbrace{e^{-2t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t e^{-2t} \begin{bmatrix} y_0 \\ 0 \end{bmatrix}}$$

So perhaps... Let $\vec{Y}_2(t) = e^{-2t} \vec{u} + t e^{-2t} \vec{v}$

$\vec{v}$ is an eigenvector of A.

But... what is $\vec{u}$?

In general... if $\boxed{\vec{y}_1 = e^{\lambda t}\vec{v}}$ solves $\vec{y}\,' = A\vec{y}$ and $\boxed{\vec{y}_2 = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{u}}$ also solves $\vec{y}\,' = A\vec{y}$, what is $\vec{u}$?

Calculate $\vec{y}_2\,'$ & $A\vec{y}_2$:

$$\vec{y}_2\,' = e^{\lambda t}\vec{v} + \lambda t e^{\lambda t}\vec{v} + \lambda e^{\lambda t}\vec{u}$$

$$A\vec{y}_2 = Ate^{\lambda t}\vec{v} + Ae^{\lambda t}\vec{u}$$

these are equal.

$e^{\lambda t}$: $\vec{v} + \lambda\vec{u} = A\vec{u}$ $\boxed{(A-\lambda I)\vec{u} = \vec{v}}$

$te^{\lambda t}$: $\lambda\vec{v} = A\vec{v} \to (A-\lambda I)\vec{v} = \vec{0}$

$\vec{u}$ is the vector that solves $(A-\lambda I)\vec{u} = \vec{v}$.

(nothing new... Confirms that $\lambda, \vec{v}$ are an eigenpair)

$\vec{v}$ is an actual eigenvector.

$\vec{u}$ is called a "generalized" eigenvector.

So... back to $\vec{Y}' = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \vec{Y}$.

$\lambda = -2$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So $\vec{Y}_1(t) = e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Then $\vec{Y}_2(t) = te^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-2t} \vec{u}$,

where $(A - \lambda I)\vec{u} = \vec{v}$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$0u_1 + u_2 = 1 \Rightarrow u_2 = 1$$

u_1 = anything.

$\therefore \vec{Y}_2(t) = te^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$.

And $\vec{Y}(t) = c_1 Y_1(t) + c_2 Y_2(t)$

$$= c_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \left(te^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$= e^{-2t} \begin{bmatrix} c_1 + c_2 t \\ c_2 \end{bmatrix}$$

For higher dimensions:

k eigenvects.

$$\begin{cases} (A-\lambda I)\vec{v} = \vec{0} \\ (A-\lambda I)\vec{u}_1 = \vec{v} \\ (A-\lambda I)\vec{u}_2 = \vec{u}_1 \\ \quad\vdots \\ (A-\lambda I)\vec{u}_{k-1} = \vec{u}_{k-2} \end{cases}$$

$\vec{v}$ is an actual eigenvect.

λ has multiplicity of k...

then $\{\vec{v}, \vec{u}_1, \vec{u}_2, \dots, \vec{u}_{k-1}\}$ is a length-k chain of generalized eigenvectors.

$$\vec{x}_1 = ve^{\lambda t}$$

$$\vec{x}_2 = vte^{\lambda t} + u_1 e^{\lambda t}$$

$$\vec{x}_3 = \frac{vt^2}{2}e^{\lambda t} + u_1 t e^{\lambda t} + u_2 e^{\lambda t}$$

$$\vec{x}_4 = \frac{vt^3}{3!}e^{\lambda t} + \frac{u_1 t^2}{2}e^{\lambda t} + u_2 t e^{\lambda t} + u_3 e^{\lambda t}$$

$$\vdots$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$(\lambda - 3)^2 = 0$$

$$\lambda_{1,2} = 3.$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2 free vars, so 2-dim soln space

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;\; ; \;\; v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So $\vec{x}' = A\vec{x}$ yields

$$\vec{x}_1 = e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{x}_2 = e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X = c_1\begin{bmatrix} e^{3t} \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{3t} \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{3t} \end{bmatrix}$$

$$y = \frac{c_2}{c_1} x.$$