# Complex Eigenvalues & Eigenvectors

$$x' = -2x - 3y \qquad \vec{Y}' = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix} \vec{Y}$$
$$y' = 3x - 2y$$

Eigenvalue shortcut:

$$\det\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right] = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$
$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$
$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

<u>OR</u>: $$\boxed{\lambda^2 - T\lambda + D = 0}$$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

$$x' = -2x - 3y \qquad \vec{Y}' = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix} \vec{Y}$$
$$y' = 3x - 2y$$

$$\lambda^2 - (-4)\lambda + 13 = 0$$
$$(\lambda + 2)^2 + 9 = 0$$
$$\boxed{\lambda_{1,2} = -2 \pm 3i}$$

So $\vec{Y}_1(t) = e^{(-2+3i)t}\vec{v}_1$ $\quad \cancel{\vec{Y}_2(t) = e^{(-2-3i)t}\vec{v}_2}$

(for real sols ignore one of these)

For $\lambda = -2+3i$:

$$\begin{bmatrix} -2-(-2+3i) & -3 \\ 3 & -2-(-2+3i) \end{bmatrix} \vec{v}_1 = \vec{0}$$

$$\begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix} \vec{v}_1 = \vec{0} \Rightarrow \boxed{\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}.}$$

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So $\vec{Y}(t) = e^{(-2+3i)t}\begin{bmatrix} i \\ 1 \end{bmatrix}$

$$= e^{-2t}(\cos 3t + i \sin 3t)\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$= e^{-2t}\begin{bmatrix} i\cos 3t - \sin 3t \\ \cos 3t + i \sin 3t \end{bmatrix}$$

$$\vec{Y}(t) = \underbrace{e^{-2t}\begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix}}_{\vec{Y}_{Re}(t)} + i\underbrace{e^{-2t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix}}_{\vec{Y}_{Im}(t)}$$

We have a theorem that tells us that $\vec{Y}_{Re}(t)$ & $\vec{Y}_{Im}(t)$ are lin. ind. sols. to the system!

So $\vec{Y}_{Re}(t) = e^{-2t}\begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix}$

$\vec{Y}_{Im}(t) = e^{-2t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix}$

And $\vec{Y}(t) = k_1 e^{-2t}\begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix} + k_2 e^{-2t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix}$

$$= \begin{bmatrix} e^{-2t}(-k_1 \sin 3t + k_2 \cos 3t) \\ e^{-2t}(k_1 \cos 3t + k_2 \sin 3t) \end{bmatrix}$$

$\lambda = -2 \pm 3i$

$K_1 \cos(3t - \alpha)$

$K_2 \cos(3t - \alpha)$ ← we could go this far..

$$\begin{cases} x' = 2y \\ y' = -3x + 2y \end{cases} \qquad \vec{Y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{Y}'(t) = \begin{bmatrix} 0 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\lambda^2 - 2\lambda + 6 = 0$$

Recall: $\lambda^2 - T\lambda + D = 0$

$$(\lambda - 1)^2 + 5 = 0$$

$$\boxed{\lambda_{1,2} = 1 \pm i\sqrt{5}}$$

For $\lambda_1 = 1 + i\sqrt{5}$:

$$\begin{bmatrix} 0-(1+i\sqrt{5}) & 2 \\ -3 & 2-(1+i\sqrt{5}) \end{bmatrix} \vec{v}_1 = \vec{0}$$

$$\begin{bmatrix} -1-i\sqrt{5} & 2 \\ -3 & 1-i\sqrt{5} \end{bmatrix} \vec{v}_1 = \vec{0}$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1+i\sqrt{5} \end{bmatrix}$$

$$\text{So } \vec{Y_1}(t) = e^{\lambda_1 t}\vec{v}_1 = \underbrace{e^{(1+i\sqrt{5})t} \begin{bmatrix} 2 \\ 1+i\sqrt{5} \end{bmatrix}}$$

(this is a solution, but it's complex-valued ☹)

$$e^{(1+i\sqrt{5})t} = e^t(\cos\sqrt{5}t + i\sin\sqrt{5}t).$$

$$\text{and } \vec{Y}_1(t) = e^t(\cos\sqrt{5}t + i\sin\sqrt{5}t) \begin{bmatrix} 2 \\ 1+i\sqrt{5} \end{bmatrix}$$

(***) key step!!

$$= \begin{bmatrix} e^t(2\cos\sqrt{5}t + 2i\sin\sqrt{5}t) \\ e^t(\cos\sqrt{5}t + i\sin\sqrt{5}t + i\sqrt{5}\cos\sqrt{5}t - \sqrt{5}\sin\sqrt{5}t) \end{bmatrix}$$

We have a theorem that tells us that whenever $Y = a(t) + b(t)i$ is a solution, then $a(t)$ and $b(t)$ are lin. ind. solutions!

$$\vec{Y}_{Re}(t) = \begin{bmatrix} 2e^t\cos\sqrt{5}t \\ e^t(\cos\sqrt{5}t - \sqrt{5}\sin\sqrt{5}t) \end{bmatrix}$$

$$\vec{Y}_{Im}(t) = \begin{bmatrix} 2e^t\sin\sqrt{5}t \\ e^t(\sin\sqrt{5}t + \sqrt{5}\cos\sqrt{5}t) \end{bmatrix}$$

each of these are real-valued solutions, and they're lin. ind.

$$\vec{Y}(t) = k_1\vec{Y}_{Re}(t) + k_2\vec{Y}_{Im}(t)$$

$$= k_1 e^t \begin{bmatrix} 2\cos\sqrt{5}t \\ \cos\sqrt{5}t - \sqrt{5}\sin\sqrt{5}t \end{bmatrix} + k_2 e^t \begin{bmatrix} 2\sin\sqrt{5}t \\ \sin\sqrt{5}t + \sqrt{5}\cos\sqrt{5}t \end{bmatrix}$$

$$= \begin{bmatrix} e^t(2k_1\cos\sqrt{5}t + 2k_2\sin\sqrt{5}t) \\ e^t((k_1 + k_2\sqrt{5})\cos\sqrt{5}t + (-k_1\sqrt{5} + k_2)\sin\sqrt{5}t) \end{bmatrix}$$

$$= \begin{bmatrix} Ce^t\cos(\sqrt{5}t - \alpha) \\ Ke^t\cos(\sqrt{5}t - \alpha_2) \end{bmatrix} \quad \text{(if needed)}$$

$$= k_1 e^t \begin{bmatrix} 2\cos\sqrt{5}t \\ \cos\sqrt{5}t - \sqrt{5}\sin\sqrt{5}t \end{bmatrix} + k_2 e^t \begin{bmatrix} 2\sin\sqrt{5}t \\ \sin\sqrt{5}t + \sqrt{5}\cos\sqrt{5}t \end{bmatrix}$$

Now, with $\vec{Y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ \sqrt{5} \end{bmatrix} = \begin{bmatrix} 2k_1 \\ k_1 + \sqrt{5}k_2 \end{bmatrix}$$

$$k_1 = \frac{1}{2}, \quad k_2 = \frac{1}{2\sqrt{5}}$$

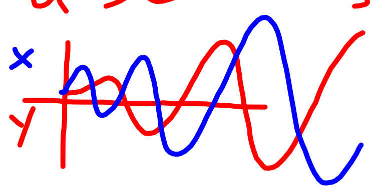
$$\vec{Y}(t) = \frac{1}{2} e^t \begin{bmatrix} 2\cos\sqrt{5}t \\ \cos\sqrt{5}t - \sqrt{5}\sin\sqrt{5}t \end{bmatrix} + \frac{1}{2\sqrt{5}} e^t \begin{bmatrix} 2\sin\sqrt{5}t \\ \sin\sqrt{5}t + \sqrt{5}\cos\sqrt{5}t \end{bmatrix}$$

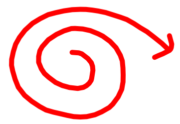
If eigenvalues of a planar (2-dim) system are complex $\boxed{\lambda = a \pm bi}$ :

freq. of oscillations

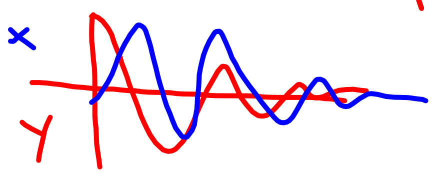
in? out? center?

If $a > 0$ — spiral source



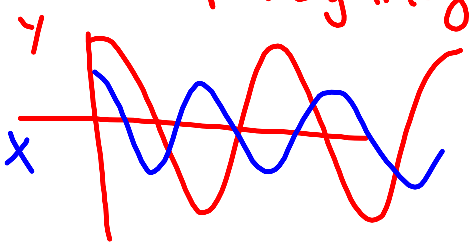



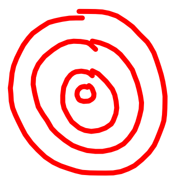
If $a < 0$ — spiral sink




If $a = 0$ — center

(purely imaginary)



or