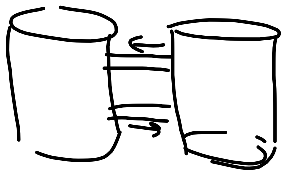
Systems & Eigenvalues/vectors

Recall:

$$\vec{Y}' = A\vec{Y}$$

OR
$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

OR
$$\begin{cases} x'(t) = ax(t) + by(t) \\ y'(t) = cx(t) + dy(t) \end{cases}$$



Assume solutions are of the form

$$\vec{X} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} ue^{\lambda t} \\ ve^{\lambda t} \end{bmatrix}$$

(similar to $ay'' + by' + cy = 0$. Let $y = e^{rt} \ldots$)

$\vec{X}' = A\vec{X}$ becomes:

$$\vec{X}' = \begin{bmatrix} \lambda u e^{\lambda t} \\ \lambda v e^{\lambda t} \end{bmatrix} \quad \text{and} \quad A\vec{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} ue^{\lambda t} \\ ve^{\lambda t} \end{bmatrix}$$

$$\vec{X}' = A\vec{X}$$

$$\lambda \cancel{e^{\lambda t}} \begin{bmatrix} u \\ v \end{bmatrix} = \cancel{e^{\lambda t}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\boxed{\lambda \vec{v} = A\vec{v}} \Rightarrow \text{solution is } e^{\lambda t}\begin{bmatrix} u \\ v \end{bmatrix}.$$

but this is the equation that leads us to eigenvalues & eigenvectors.

Eigenvalues __can__ be zero, but eigenvectors __cannot__ be zero.

Ex:

$$\vec{Y}' = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \vec{Y}$$

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\begin{cases} x'(t) = -3x(t) \\ y'(t) = 2y(t). \end{cases}$$

x' only involves x, y' only involves y... We call this a "decoupled" system.

$$\boxed{\begin{aligned} x(t) &= c_1 e^{-3t} \\ y(t) &= c_2 e^{2t} \end{aligned}}$$

$$\vec{Y}' = A\vec{Y} \qquad Y' = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \vec{Y}$$

Use eigenvalues & eigenvectors:

$$\begin{vmatrix} -3-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(-3-\lambda)(2-\lambda) - 0 = 0$$

$$\boxed{\lambda_1 = -3 \quad \lambda_2 = 2}$$

note: eigenvalues of a diagonal matrix are the diagonal entries!!

For $\lambda_1 = -3$

$$\begin{bmatrix} -3+3 & 0 \\ 0 & 2+3 \end{bmatrix} \vec{V}_1 = \vec{0}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \vec{V}_1 = \vec{0} \implies \vec{V}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

From λ_1 and $\vec{V}_1$ we obtain $\vec{Y}_1(t) = e^{-3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For $\lambda_2 = 2$:

$$\begin{bmatrix} -3-2 & 0 \\ 0 & 2-2 \end{bmatrix} \vec{V}_2 = \vec{0} \implies \vec{V}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

From λ_2 and $\vec{V}_2$ we obtain $\vec{Y}_2(t) = e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\therefore \vec{Y}(t) = c_1 \vec{Y}_1(t) + c_2 \vec{Y}_2(t)$$

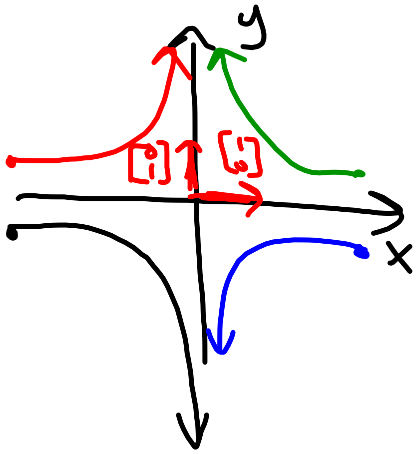
$$= c_1 e^{-3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{-3t} + 0 \\ 0 + c_2 e^{2t} \end{bmatrix}$$

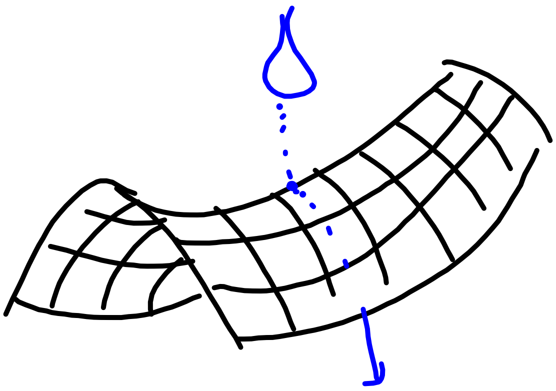
$$\boxed{\vec{Y}(t) = \begin{bmatrix} c_1 e^{-3t} \\ c_2 e^{2t} \end{bmatrix}}$$ (same as above)

$$\vec{Y}' = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \vec{Y} \Rightarrow \boxed{\vec{Y}(t) = \begin{bmatrix} c_1 e^{-3t} \\ c_2 e^{2t} \end{bmatrix}} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- as $t \to \infty$ $\quad x(t) \to 0$
  $y(t) \to \pm\infty$
- as $t \to -\infty$ $\quad x(t) \to \pm\infty$
  $y(t) \to 0$



Here, $(x, y) = (0, 0)$ is an equilibrium called a "saddle."



(unstable equilibrium)

Ex 2:

$$\vec{Y}' = \begin{bmatrix} 8 & -11 \\ 6 & -9 \end{bmatrix} \vec{Y}$$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc$$

$$ad + \lambda^2 - (a+d)\lambda - bc$$

$$\lambda^2 - (a+d)\lambda + ad - bc$$

$$\lambda^2 - (\mathrm{Tr}A)\lambda + \mathrm{Det}A$$

$$\boxed{\lambda^2 - T\lambda + D = 0}$$

useful in WebWork.

$$\lambda^2 + \lambda + (-72+66) = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0$$

$$\boxed{\lambda_1 = -3, \quad \lambda_2 = 2}$$

For $\lambda_1 = -3$:

$$\begin{bmatrix} 8+3 & -11 \\ 6 & -9+3 \end{bmatrix} \vec{v}_1 = \vec{0}$$

$$\begin{bmatrix} 11 & -11 \\ 6 & -6 \end{bmatrix} \vec{v}_1 = \vec{0} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So $\vec{X_1(t)} = e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = 2$:

$$\begin{bmatrix} 6 & -11 \\ 6 & -11 \end{bmatrix} \vec{v}_2 = \vec{0} \Rightarrow \vec{v}_2 = \begin{bmatrix} 11 \\ 6 \end{bmatrix}$$

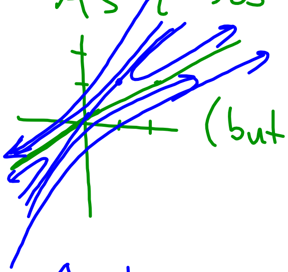
$$\vec{X_2(t)} = e^{2t}\begin{bmatrix} 11 \\ 6 \end{bmatrix}$$

So $\boxed{\vec{X(t)} = c_1 e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 11 \\ 6 \end{bmatrix}}$.

or $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-3t} + 11 c_2 e^{2t} \\ c_1 e^{-3t} + 6 c_2 e^{2t} \end{bmatrix}$

As $t \to \infty$ $\quad x \to 11 c_2 e^{2t} \to \pm\infty$

$\quad y \to 6 c_2 e^{2t} \to \pm\infty$



(but note that $y \approx \frac{6}{11} x$.)

As $t \to -\infty$ $\quad x \to c_1 e^{-3t} \to \pm\infty$

$\quad y \to c_1 e^{-3t} \to \pm\infty$

(note that $y \approx x$)

Ex:

$\vec{Y}' = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \vec{Y}$ ....

Decoupled!

$x(t) = c_1 e^{-t}$

$y(t) = c_2 e^{-4t}$

$\lambda^2 + 5\lambda + 4 = 0$

$(\lambda + 4)(\lambda + 1) = 0$

$\lambda_1 = -4, \ \lambda_2 = -1$

For $\lambda_1 = -4$:

$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \vec{v}_1 = \vec{0} \Rightarrow \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

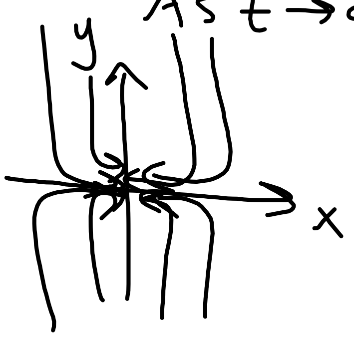
For $\lambda_2 = -1$

$\begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \vec{v}_2 = \vec{0} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{Y}(t) = c_1 e^{-4t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 e^{-t} \\ c_1 e^{-4t} \end{bmatrix}$

As $t \to \infty$, $x \to 0$.

$y \to 0$... but faster than x.



[The origin is an equilibrium* and in this case is called a "sink".]

* b/c if $(x, y) = (0, 0)$,

then $x' = -x = 0$

and $y' = -4y = 0$ ...nothing moves.

$$Y' = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix} Y$$

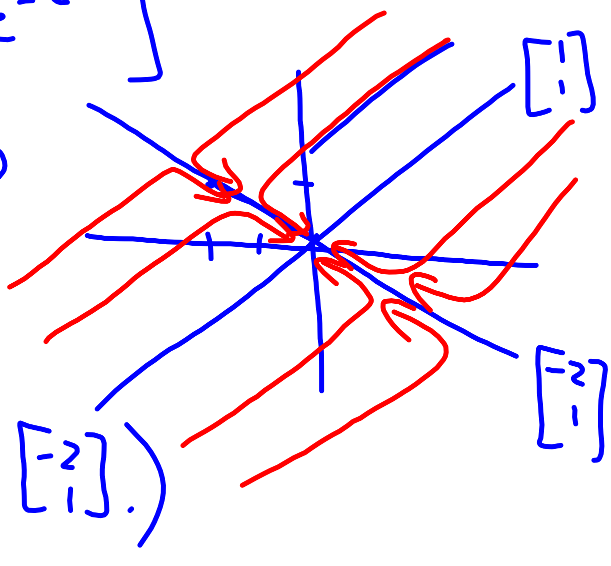
Do stuff

$$Y = \begin{bmatrix} k_1 e^{-4t} - 2k_2 e^{-t} \\ k_1 e^{-4t} + k_2 e^{-t} \end{bmatrix} = k_1 e^{-4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2 e^{-t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

As $t \to \infty$ $\quad x \to 0$

$\quad y \to 0$

(but faster along $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ than along $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.)



$$\vec{Y}' = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix} \vec{Y}$$

$$\longrightarrow k_1 e^{-4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2 e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Find the solution passing through

$Y_0 = (2,1)$

$$k_1 \cdot 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2 \cdot 1 \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$k_1 - 2k_2 = 2$$

$$k_1 + k_2 = 1$$

$$-3k_2 = 1$$

$$k_2 = -1/3$$

$$k_1 = 4/3$$

So $\vec{Y}(t) = \frac{4}{3} e^{-4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(-\frac{1}{3}\right) e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 4/3\, e^{-4t} + 2/3\, e^{-t} \\ 4/3\, e^{-4t} - 1/3\, e^{-t} \end{bmatrix}$$

In a "planar" (2-dimensional) system:

- If λ_1, λ_2 have opp. signs, origin is an unstable equilibrium called a <u>saddle</u>.
- If $\lambda_1 > \lambda_2 > 0$, origin is an unstable equilibrium called a <u>source</u>. Trajectories travel outward from near the origin along $\vec{v}_2$ but ultimately end up parallel to $\vec{v}_1$.
- If $\lambda_1 < \lambda_2 < 0$ origin is a stable equilibrium called a <u>sink</u>. Solutions approach the origin parallel to $\vec{v}_1$ but end up parallel to $\vec{v}_2$ as they get close to the origin.