Variation of Parameters

So... what if we have variable coefficients or an "unruly" forcing term (i.e., not poly, sine or cosine, or exponential)?

Start with the nonhomogeneous eqn

$$y'' + p(t)y' + q(t)y = f(t) \qquad (1)$$

Begin the same way, i.e., find $y_h(t)$.

$$\boxed{\left(\text{So } y_h(t) = c_1 y_1 + c_2 y_2\right)}$$

New idea: Instead of assuming y_p looks like $f(t)$, we assume y_p looks like y_h, but with functions as coefficients instead of constants.

Assume $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$.

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(we'll just write $y_p(t) = v_1 y_1 + v_2 y_2$)

$$y_p'(t) = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$$

(Since the v_1 & v_2 are already arbitrary, we'll impose the condition that $v_1' y_1 + v_2' y_2 = 0$.)

Then $y_p'(t) = v_1 y_1' + v_2 y_2'$.

And $y_p''(t) = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$

So eq (1) becomes

$$y'' + p(t)y' + g(t)y = f(t)$$

$$(v_1' y_1' + \cancel{v_1 y_1''} + v_2' y_2' + \cancel{v_2 y_2''}) + p(t)(\cancel{v_1} y_1' + \cancel{v_2} y_2') + g(t)(\cancel{v_1} y_1 + \cancel{v_2} y_2) = f(t)$$

$$v_1\underbrace{(y_1'' + p(t)y_1' + g(t)y_1)}_{=0} + v_2\underbrace{(y_2'' + p(t)y_2' + g(t)y_2)}_{=0} + v_1'y_1' + v_2'y_2' = f(t)$$

(b/c y_1 solves homogeneous eqn) (b/c y_2 solves homogeneous eqn)

So we're left with:

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ v_1' y_1' + v_2' y_2' = f(t) \end{cases}$$

a system we can use to solve for v_1' & v_2'.

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ v_1' y_1' + v_2' y_2' = f(t) \end{cases}$$

Start here ...

Cramer's Rule:

$$v_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ f(t) & y_2'(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} = \frac{-y_2 f}{W(y_1, y_2)}$$

$$v_2'(t) \quad \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & f(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} = \frac{y_1 f}{W(y_1, y_2)}$$

$$\Longrightarrow \boxed{\begin{aligned} v_1(t) &= \int \frac{-y_2 f}{W(y_1, y_2)}\, dt \\ v_2(t) &= \int \frac{y_1 f}{W(y_1, y_2)}\, dt \end{aligned}}$$

Example:

$$y'' + y = \sec t$$

$$r^2 + 1 = 0$$

$$r = \pm i$$

$$\boxed{y_h(t) = c_1 \cos t + c_2 \sin t}$$

VoP:

$$y_p(t) = v_1(t)\cos t + v_2(t)\sin t.$$

Recall: $\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ v_1' y_1' + v_2' y_2' = f(t) \end{cases}$

So here: $\begin{cases} v_1' \cos t + v_2' \sin t = 0 \\ -v_1' \sin t + v_2' \cos t = \sec t \end{cases}$

$$v_1' = \frac{-\sin t \sec t}{\cos^2 t + \sin^2 t} = -\tan t. \qquad \frac{\begin{vmatrix} 0 & \sin t \\ \sec t & \cos t \end{vmatrix}}{W}$$

$$v_2' = \frac{1}{\cos^2 t + \sin^2 t} = 1. \qquad \frac{\begin{vmatrix} \cos t & 0 \\ -\sin t & \sec t \end{vmatrix}}{W}$$

So $v_1 = \int -\tan t \, dt = -\ln|\sec t|$

$$v_2 = \int dt = t.$$

$$\boxed{\begin{aligned} y_p(t) &= v_1 y_1 + v_2 y_2 \\ &= -\ln|\sec t|\cos t + t \sin t \end{aligned}}$$

particular solution

FINALLY:

$$y(t) = y_h(t) + y_p(t)$$

$$= c_1 \cos t + c_2 \sin t + (-\ln|\sec t|)\cos t + t \sin t.$$