5.5 – Nonhomogeneous Eqns

$$ma = \sum F$$

$$mx'' = -kx - cx' + f(t)$$

$$\boxed{mx'' + cx' + kx = f(t)}$$

$m > 0$ (mass)

$k > 0$ (spring)

$c \geq 0$ (damping)
(undamped if $c = 0$)

$f(t)$ = external forcing function

(we call this "forced" or "driven" harmonic motion).

5.5 - Nonhomogeneous Eqns

Two methods:

- Undetermined Coefficients
- Variation of Parameters

→ works for forcing functions that are polynomial, trig, or exponential (or combinations of these).

Ex 1

Solve $y'' + 3y' + 4y = 3x+2$ → $f(x) \neq 0$ (non-homogeneous)

(1) Solve the associated homogeneous equation.

$$y'' + 3y' + 4y = 0.$$

$$r^2 + 3r + 4 = 0$$

$$\left(r + \tfrac{3}{2}\right)^2 + \tfrac{7}{4} = 0$$

$$\Rightarrow r = -\tfrac{3}{2} \pm i\tfrac{\sqrt{7}}{2}$$

$$y_h = e^{-\frac{3}{2}x}\left(c_1\cos\left(\tfrac{\sqrt{7}}{2}x\right) + c_2\sin\left(\tfrac{\sqrt{7}}{2}x\right)\right)$$

→ the "homogeneous" or the "complimentary" solution.

(2) Find a "particular solution" y_p by "guessing" an appropriate y_p and using "undetermined coefficients".

$f(x) = 3x+2$, so let $y_p = Ax + B$.

$y_p' = A$

$y_p'' = 0$

$y'' + 3y' + 4y = 3x+2$ becomes

$$(0) + 3(A) + 4(Ax+B) = 3x+2$$

y_p'' $3y_p'$ $4y_p$

x: $4A = 3$ $A = 3/4$

1: $3A + 4B = 2$ $B = -1/16$

So $y_p = \frac{3}{4}x - \frac{1}{16}$

(3) Construct the "general solution":

$$y(x) = y_h(x) + y_p(x)$$

$$y(x) = \underbrace{e^{-\frac{3}{2}x}\left(c_1\cos\left(\tfrac{\sqrt{7}}{2}x\right) + c_2\sin\left(\tfrac{\sqrt{7}}{2}x\right)\right)}_{y_h} + \underbrace{\tfrac{3}{4}x - \tfrac{1}{16}}_{y_p}.$$

y_h: solves all IVP (if we include y_p).

y_p: solves only $y(0) = -1/16$.

(4) Solve for c_1 & c_2 by using the initial conditions.

Ex 2:

Solve $y'' - 4y = 2e^{3x}$

- Solve for y_h: $r^2 - 4 = 0$

  $r = \pm 2$

  $y_h = c_1 e^{-2x} + c_2 e^{2x}$

- Solve for y_p:

  Let $y_p = Ae^{3x}$

  $y_p' = 3Ae^{3x}$

  $y_p'' = 9Ae^{3x}$

  $y_p'' - 4y_p = 9Ae^{3x} - 4(Ae^{3x}) = 2e^{3x}$

  $5A = 2$

  $A = 2/5$

- $y = y_h + y_p = c_1 e^{-2x} + c_2 e^{2x} + \frac{2}{5} e^{3x}$

- Init. cond., if any?

Ex 3:

$3y'' + y' - 2y = 2\cos x$

y_h: $(3r-2)(r+1)=0$

$$\boxed{y_h = c_1 e^{\frac{2}{3}x} + c_2 e^{-x}}$$

y_p: Let $y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

$3y_p'' + y_p' - 2y_p = 2\cos x$ becomes:

$3(-A\cos x - B\sin x) + (-A\sin x + B\cos x)$
$-2(A\cos x + B\sin x) = 2\cos x$

$\cos x$: $-3A + B - 2A = 2 = -5A + B$

$\sin x$: $-3B - A - 2B = 0 = -A - 5B$

$A = -5B$

$2 = 26B$

$\frac{1}{13} = B$

$A = -\frac{5}{13}$

So

$y_p = -\frac{5}{13}\cos x + \frac{1}{13}\sin x$

- $y = y_h + y_p$

$= c_1 e^{\frac{2}{3}x} + c_2 e^{-x} - \frac{5}{13}\cos x + \frac{1}{13}\sin x.$

Ex 4:

$y'' + 4y = 3x^3$

$r^2 + 4 = 0$

$r = \pm 2i$

$y_h = c_1 \cos 2x + c_2 \sin 2x$

$y_p = Ax^3 + Bx^2 + Cx + D$ (need all powers of x here)

$y_p' = 3Ax^2 + 2Bx + C$

$y_p'' = 6Ax + 2B$

Ex 5:

$y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$

$y_h = c_1 e^{2x} + c_2 e^{x}$

Let $y_p = Ae^{-x} + B\cos 3x + C\sin 3x$

⋮

etc.

EX 6: (AVOID DUPLICATION)

$$y'' - 4y = 2e^{2x}$$

Solve for y_h:

$r^2 - 4 = 0$

$r = \pm 2 \Longrightarrow y_h = c_1 e^{-2x} + c_2 e^{2x}$.

Solve for y_p:

Let $y_p = Ae^{2x}$ — our initial guess would duplicate some or all of y_h.

~~$y_p' = 2Ae^{2x}$~~

~~$y_p'' = 4Ae^{2x}$~~

~~So $y'' - 4y = 2e^{2x}$ becomes~~

~~$4Ae^{2x} - 4(Ae^{2x}) = 2e^{2x}$.~~

~~$0A = 2$. ?!?~~

Instead multiply guess by x^s, where s is the smallest integer necessary to avoid all duplication.

$y_p = Ae^{2x}$ duplicates $c_1 e^{2x}$ from y_h.

So let $y_p = Axe^{2x}$.

$y_p' = Ae^{2x} + 2Axe^{2x}$

$y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$

$$y'' - 4y = \underbrace{(4Ae^{2x} + 4Axe^{2x})}_{y_p''} \underbrace{- 4(Axe^{2x})}_{-4y_p} = \underbrace{2e^{2x}}_{f(x)}$$

$4A = 2$

$A = \frac{1}{2}$.

So $y = y_h + y_p = c_1 e^{-2x} + c_2 e^{2x} + \frac{1}{2}xe^{2x}$.

Ex 7:

$y^{(3)} + 9y' = x\sin x + x^2 e^{2x}$

$r^3 + 9r = 0$

$r(r^2+9) = 0$

$r = 0, \quad r = \pm 3i$

$$\boxed{y_h = c_1 + c_2\cos(3x) + c_3\sin(3x)}$$

$y_p = A\cos x + B\sin x + Cx\cos x + Dx\sin x$
$\quad + Ee^{2x} + Fxe^{2x} + Gx^2e^{2x}$

~~$(Ax+B)(\cos x + \sin x) + (C + Dx + Ex^2)e^{2x}$~~

$(Ax+B)(C\cos x + D\sin x) + (C + Dx + Ex^2)e^{2x}$

$\alpha x\cos x + \beta x\sin x + \gamma\cos x + \delta\sin x$

Ex 7b

$f(x) = x^2 e^{2x} + x \sin 3x$

$(r-2)^3 (r^2+9) = 0$

$y_h = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}$

$\quad + c_4 \cos 3x + c_5 \sin 3x$

First:

$y_p = (A + Bx + Cx^2) e^{2x} x^3$

$\quad + x \cdot \left[(Dx+E)(\cos 3x) + (Fx+G) \sin 3x \right]$

Ex 8

$$y''' + y'' = 3e^x + 4x^2$$

$$r^3 + r^2 = r^2(r+1) = 0$$

$r = 0, 0, -1$

$y_1 = e^{0x} = 1 \quad y_2 = xe^{0x} = x \quad y_3 = e^{-x}$

$$\Rightarrow \quad y_h = c_1 + c_2 x + c_3 e^{-x}.$$

$$y_p = Ae^x + (Bx^2 + Cx + D)x^2$$

Multiply the part of the solution that duplicates y_h by x^s, where s is the smallest integer needed to avoid duplication.

Ex 9

$$y'' + 6y' + 13y = e^{-3x}\cos 2x$$

$$(r+3)^2 + 4 = 0$$

$$r = -3 \pm 2i$$

$$\hookrightarrow y_h = e^{-3x}(c_1\cos 2x + c_2 \sin 2x)$$

$$y_p = e^{-3x}(c_1\cos 2x + c_2 \sin 2x)x$$