5.5 - NON-homogeneous 2nd-order Differential Equations

Consider the equation

$$y'' + p(t)y' + q(t)y = g(t) \qquad (1)$$

and its "associated homogeneous equation"

$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

THEOREM: If $Y_1(t)$ and $Y_2(t)$ are both solutions to (1) then their difference $Y_1(t) - Y_2(t)$ is a solution to (2), and can therefore be written as $Y_1(t) - Y_2(t) = c_1y_1(t) + c_2y_2(t)$, where $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions to (2).

<u>PROOF:</u>

Recall

$$y'' + p(t)y' + q(t)y = g(t) \qquad (1)$$

$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

Substitute $Y_1(t) - Y_2(t)$ into (2).

$$(Y_1 - Y_2)'' + p(t)(Y_1 - Y_2)' + q(t)(Y_1 - Y_2)$$

$$= Y_1'' - Y_2'' + p(t)Y_1' - p(t)Y_2' + q(t)Y_1 - q(t)Y_2$$

$$= \underbrace{\left(Y_1'' + p(t)Y_1' + q(t)Y_1\right)}_{g(t)} - \underbrace{\left(Y_2'' + p(t)Y_2' + q(t)Y_2\right)}_{g(t)}$$

$$= g(t) - g(t)$$

$$= 0.$$

$\therefore$ $Y_1 - Y_2$ is a soln to the associated homogeneous eqn.

How is the THEOREM helpful? Recall (1) and (2):

$$y'' + p(t)y' + q(t)y = g(t) \qquad (1)$$

$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

If $y(t)$ is the ***general solution*** to (1) and $y_p(t)$ is ***any*** solution to (1), then by the theorem we know that

$$y(t) - y_p(t) = c_1y_1(t) + c_2y_2(t)$$

but this implies that the ***general solution*** to (1) can therefore be expressed as

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t).$$

(We tend to set $c_1y_1(t) + c_2y_2(t) = y_h(t)$ or $y_c(t)$ to indicate that it is the solution to the associated homogeneous equation (hence the y_h), also called the "complementary" solution (hence the y_c).)

Therefore, given

$$y'' + p(t)y' + q(t)y = g(t) \qquad (1)$$

$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

the general solution to (1) is given by

$$y(t) = y_h(t) + y_p(t),$$

where $y_p(t)$ is *any* particular solution to (1) and $y_h(t)$ is the *general solution* to (2) (i.e., the solution to the associated homogeneous equation to (1)).

WHY do we need both?

* $y_p(t)$ allows us to address the nonhomogeneous g(t) term, but there are typically no constants of integration to allow for different initial conditions - we would not have the **general** solution to (1).

* $y_h(t)$, on the other hand, contains constants of integration that allow us to handle different initial conditions, but cannot address the nonhomogeneous term (since it only solves the equation in which g(t) = 0) - we would not even have an actual solution to (1).

So ... we have had lots of practice finding solutions $y_h(t)$. How do we find the particular solutions $y_p(t)$?

Two common methods:

1) Undetermined coefficients

2) Variation of parameters

Each has strengths and limitations. We will start with the method of "Undetermined Coefficients," which works really well with constant coefficients and nonhomogeneous terms that are polynomials, sines and cosines, exponential functions, or sums and products thereof.

EXAMPLE 1: Solve y'' - 3y' - 10y = $4e^{3t}$

(1) Find $y_h(t)$; (i.e., solve $y''-3y'-10y=0$)

$r^2-3r-10=0$

$(r-5)(r+2)=0$

$$\boxed{y_h(t)=c_1e^{5t}+c_2e^{-2t}.}$$

(2) Find $y_p(t)$.

Assume $y_p(t)=Ae^{3t}$

$y_p' = 3Ae^{3t}$

$y_p'' = 9Ae^{3t}$

if $y=Ae^{3t}$ is a solution, this should equal

$y_p''-3y_p'-10y_p = 9Ae^{3t}-3(3Ae^{3t})-10(Ae^{3t}) = 4e^{3t}$

$-10Ae^{3t} = 4e^{3t}$

$-10A=4$

$$\boxed{A=-\frac{2}{5}.}$$

$\therefore\ y_p(t)=-\frac{2}{5}e^{3t}.$

$y(t)=y_h(t)+y_p(t)$

$$\boxed{y(t)=c_1e^{5t}+c_2e^{-2t}-\frac{2}{5}e^{3t}.}$$

EXAMPLE 2: Solve y'' - 3y' - 10y = 2cos(3t)

Same LHS: $y_h(t) = c_1 e^{5t} + c_2 e^{-2t}$

Guess for y_p: $\boxed{y_p = A\cos 3t + B\sin 3t.}$

EXAMPLE 3: Solve y'' - 3y' - 10y = $2t^3$

$$y_h(t) = c_1 e^{5t} + c_2 e^{-2t}$$

$$y_p(t) = At^3 + Bt^2 + Ct + D.$$

Generalizing thus far:

$g(t)$	$Y_P(t)$ **guess**
$a\mathbf{e}^{\beta t}$	$A\mathbf{e}^{\beta t}$
$a\cos(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
n^{th} degree polynomial	$A_n t^n + A_{n-1}t^{n-1} + \cdots A_1 t + A_0$

EXAMPLE 4: Solve $y'' - 3y' - 10y = 3t^2e^{4t}$

Same y_h.

y_p: $\quad y_p = e^{4t}(At^2 + Bt + C)$

EXAMPLE 4A: $\quad y'' - 3y' - 10y = e^{4t}(\cos 3t)$

$y_p = e^{4t}(A\cos 3t + B\sin 3t)$

<u>Guessing Practice:</u> For each of the following, write the form of the particular solution to the equation

$$y'' + p(t)y' + q(t) = g(t),$$

but DO NOT solve the equation.

(a) $g(t) = (10 - 3t)e^{2t}$

$y_p = (At+B)e^{2t}$

(b) $g(t) = (3t^2 + 4)\sin(3t)$

$(At^2+Bt+C)\sin 3t + (Dt^2+Et+F)\cos 3t$

(c) $g(t) = 2e^{2t}\cos(3t)$

$y_p = e^{2t}(A\cos 3t + B\sin 3t)$

(d) $g(t) = -3e^{-2t}\cos(5t)(3 - t^3)$

(d) $e^{-2t}(At^3+Bt^2+Ct+D)\cos(5t)$
$+ e^{-2t}(Et^3+Ft^2+Gt+H)\sin(5t)$

EXAMPLE 5: Solve $y'' - 3y' - 10y = 2t^3 + 4e^{3t} + 2\cos(3t)$

$y_h(t) = c_1 e^{5t} + c_2 e^{-2t}$

$y_{p_1}(t)$ is for $2t^3$

$y_{p_2}(t)$ is for $4e^{3t}$

$y_{p_3}(t)$ is for $2\cos 3t$

So $y_p = y_{p_1} + y_{p_2} + y_{p_3}$.

The previous example shows that if $y_p(t)$ is a solution to

$$y'' + p(t)y' + q(t)y = f(t)$$

and $y_q(t)$ is a solution to

$$y'' + p(t)y' + q(t)y = g(t),$$

then $y_p(t) + y_q(t)$ is a solution to

$$y'' + p(t)y' + q(t)y = f(t) + g(t).$$

<u>EXAMPLE</u>: $y'' - 3y' - 10y = 4e^{3t} + 2t$

$y_p = Ae^{3t}$ $\quad\quad$ $y_q = At + B$

$\vdots$ $\quad\quad$ $\vdots$

<u>OR</u> $y_p = Ae^{3t} + Bt + C$

<u>More Guessing Practice:</u> For each of the following, write the form of the particular solution to the equation y'' + p(t)y' + q(t) = g(t).

(a) $g(t) = 4\cos(6t) - 9\sin(6t)$

$y_p = A\cos 6t + B\sin 6t$

(b) $g(t) = -2\sin t + \sin(14t) - 5\cos(14t)$

$y_p = A\sin t + B\cos t + C\sin 14t + D\cos 14t$

(c) $g(t) = e^{7t} + 6$

$y_p = Ae^{7t} + B$

(d) $g(t) = 6t^2 - 7\sin(3t) + 9$

$At^2 + Bt + C + D\sin 3t + E\cos 3t$

(e) $g(t) = 10e^t - 5te^{-8t} + 2e^{-8t}$

$Ae^t + e^{-8t}(Bt + C)$

(f) $g(t) = t^2\cos t - 5t\sin t$

$(At^2 + Bt + C)\cos t + (Dt^2 + Et + F)\sin t$

(g) $g(t) = 5e^{-3t} + e^{-3t}\cos(6t) - \sin(6t)$

$$y_p = Ae^{-3t} + e^{-3t}(B\cos 6t + C\sin 6t) + D\sin 6t + E\cos 6t$$

EXAMPLE 6: Solve y'' - 3y' - 10y = 3e^{5t}.

$$\boxed{y_h(t) = c_1 e^{5t} + c_2 e^{-2t}.}$$

$$y_p(t) = Ate^{5t}$$

oops... avoid duplicating any solution to the homogeneous eqn.

$$y_p' = Ae^{5t} + 5Ate^{5t}$$

$$y_p'' = 5Ae^{5t} + 5Ae^{5t} + 25Ate^{5t} = 10Ae^{5t} + 25Ate^{5t}$$

$$10Ae^{5t} + 25Ate^{5t} - 3(Ae^{5t} + 5Ate^{5t}) - 10(Ate^{5t}) = 3e^{5t}$$

$$e^{5t}: \quad 10A - 3A = 3 \implies A = \frac{3}{7}.$$

$$te^{5t}: \quad 25A - 15A - 10A = 0$$

$$y(t) = c_1 e^{5t} + c_2 e^{-2t} + \frac{3}{7} te^{5t}.$$

Even More Guessing Practice: For each of the following, write the form of the particular solution to the equation

y" + p(t)y' + q(t) = g(t).

(a) $y'' + 3y' - 28y = 7t + \mathbf{e}^{-7t} - 1$

(b) $y'' - 100y = 9t^2\mathbf{e}^{10t} + \cos t - t\sin t$

(c) $4y'' + y = \mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

(d) $4y'' + 16y' + 17y = \mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

(e) $y'' + 8y' + 16y = \mathbf{e}^{-4t} + \left(t^2 + 5\right)\mathbf{e}^{-4t}$

Put it all together:

Solve y'' - 3y' + 2y = 3e^{-t} - 10cos(3t) with y(0) = 1 and y'(0) = 2.

$$(r^2-3r+2)=0$$
$$(r-2)(r-1)=0$$
$$\boxed{y_h(t)=c_1e^{2t}+c_2e^{t}}$$

$$\begin{cases} y_p = Ae^{-t}+B\cos 3t+C\sin 3t \\ y_p' = -Ae^{-t}-3B\sin 3t+3C\cos 3t \\ y_p'' = Ae^{-t}-9B\cos 3t-9C\sin 3t \end{cases}$$

$$y''-3y'+2y=3e^{-t}-10\cos 3t$$

$\underline{e^{-t}}$: $A+3A+2A=3 \Rightarrow A=\frac{1}{2}$.

$\underline{\cos 3t}$: $-9B-9C+2B=-10$

$\underline{\sin 3t}$: $-9C+9B+2C=0$

$$-7B-9C=-10$$
$$9B-7C=0$$
$$B=\frac{7C}{9}$$
$$-63B-81C=-90$$
$$63B-49C=0$$
$$-130C=-90$$
$$C=\frac{9}{13}$$
$$B=\frac{7}{13}.$$

$$\boxed{y(t)=c_1e^{2t}+c_2e^{t}+\frac{1}{2}e^{-t}+\frac{7}{13}\cos 3t+\frac{9}{13}\sin 3t}$$

$y(0)=1 \Rightarrow 1=c_1+c_2+\frac{1}{2}+\frac{7}{13}$

$y'(0)=2$: $\quad c_1+c_2=-\frac{1}{26}$

$$2=2c_1e^{2t}+c_2e^{t}-\frac{1}{2}e^{-t}-\frac{21}{13}\sin 3t+\frac{27}{13}\cos 3t\Big|_{t=0}$$

$$2=2c_1+c_2-\frac{1}{2}+\frac{27}{13}$$

$$2c_1+c_2=\frac{11}{26}$$
$$c_1+c_2=-\frac{1}{26}$$
$$c_1=\frac{12}{26}$$
$$c_2=-\frac{13}{26}$$

$$\boxed{y(t)=\frac{6}{13}e^{2t}-\frac{1}{2}e^{t}+\frac{1}{2}e^{-t}+\frac{7}{13}\cos 3t+\frac{9}{13}\sin 3t}$$