5.4 - Harmonic Motion

$$mx'' + bx' + kx = 0 \qquad m, k, b > 0$$

Undamped $(b=0)$.

$$mx'' + kx = 0$$
$$mr^2 + k = 0$$
$$r^2 = -\frac{k}{m}$$
$$r = \pm\sqrt{\frac{-k}{m}} = \pm i\sqrt{\frac{k}{m}} = 0 \pm i\sqrt{\frac{k}{m}}$$

Complex-valued sols:

$$\boxed{y_1 = e^{(0+i\sqrt{k/m})t}} \qquad y_2 = e^{(0-i\sqrt{k/m})t}$$

Real-valued sols:

$$y = e^{0t} e^{i\sqrt{k/m}\,t} = e^{0t}\Big(\underbrace{\cos\sqrt{\tfrac{k}{m}}\,t}_{Re} + \underbrace{i\sin\sqrt{\tfrac{k}{m}}\,t}_{Im}\Big)$$

$$y_1 = e^{0t}\cos\sqrt{\frac{k}{m}}\,t \qquad y_2 = e^{0t}\sin\sqrt{\frac{k}{m}}\,t$$

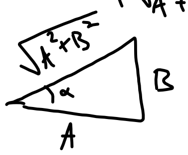
$$\boxed{\text{Let } \sqrt{\frac{k}{m}} = \omega_0}$$

Then $y = Ae^{0t}\cos(\omega_0 t) + Be^{0t}\sin(\omega_0 t)$.

or $y = A\cos(\omega_0 t) + B\sin(\omega_0 t)$.

Note: $$\frac{\sqrt{A^2+B^2}}{\sqrt{A^2+B^2}}\left(A\cos(\omega_0 t) + B\sin(\omega_0 t)\right)$$

$$= \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos(\omega_0 t) + \frac{B}{\sqrt{A^2+B^2}}\sin(\omega_0 t)\right)$$



$$= \sqrt{A^2+B^2}\left(\cos\alpha\cos\omega_0 t + \sin\alpha\sin\omega_0 t\right)$$

$$= \boxed{\sqrt{A^2+B^2}\left(\cos(\omega_0 t - \alpha)\right)}.$$

So... undamped:

$$mx'' + kx = 0 \qquad \left(\omega_0 = \sqrt{\frac{k}{m}}\right).$$

$$\vdots$$

$$x(t) = \underbrace{A\cos(\omega_0 t) + B\sin(\omega_0 t)}$$

get A, B from initial conditions.

$$= C\cos(\omega_0 t - \alpha)$$

$$C = \sqrt{A^2 + B^2}, \qquad \tan\alpha = \frac{B}{A}$$

$$0 \le \alpha < 2\pi$$

$A<0$, $B>0$	$A, B > 0$
$A, B < 0$	$A > 0$, $B < 0$

$$\boxed{x(t) = C\cos(\omega_0 t - \alpha)}$$

C : Amplitude

ω_0 : circular frequency (rad/sec)

α : phase angle (rad)

or $x(t) = C\cos\left(\omega_0\left(t - \frac{\alpha}{\omega_0}\right)\right)$

$\delta = \frac{\alpha}{\omega_0}$: time lag (sec).

ω_0 : rad/sec ...

$\frac{2\pi}{\omega_0}$: period $\frac{(\text{sec})}{\text{cycle}}$ (time it takes to complete one cycle)

$\frac{\omega_0}{2\pi}$: $\frac{\text{cycles}}{\text{sec}}$ (Hertz - Hz)

Example:

$$x'' + 100x = 0$$

characteristic eqn:

$$r^2 + 100 = 0$$

$$r = \pm\sqrt{-100} = \pm 10i$$

skipped lots.

$$x_1 = e^{0t}\cos 10t \qquad x_2 = e^{0t}\sin 10t.$$

$$\longrightarrow \boxed{x(t) = A\cos 10t + B\sin 10t.}$$

(general solution)

$$x(0) = 1, \quad x'(0) = -5.$$

$$x(0) = 1 = A\cos(0) + B\sin(0)$$

$$\Longrightarrow A = 1.$$

$$x'(0) = -5 = (-10\sin 10t + 10B\cos 10t)\Big|_{t=0}$$

$$-5 = 10B$$

$$\boxed{-\tfrac{1}{2} = B}$$

$$\boxed{x(t) = \cos(10t) - \tfrac{1}{2}\sin(10t)}$$

A=1, B=-1/2

$$C = \sqrt{1^2 + \left(-\tfrac{1}{2}\right)^2} = \frac{\sqrt{5}}{2}.$$

$$\alpha = 2\pi - \tan^{-1}\left(\frac{1/2}{1}\right) = 5.8195$$

(b/c 4th quad)

$$\text{So...} \quad x(t) = \frac{\sqrt{5}}{2}\cos(10t - 5.8195).$$

Amp: $\sqrt{5}/2$

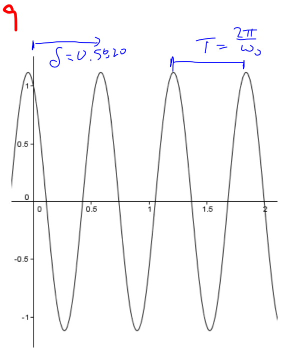
ω_0 : 10 rad/sec

α : 5.8195 rad

$T = \frac{2\pi}{\omega_0} \approx 0.6283$ sec.

$\nu = \frac{1}{T} = \frac{\omega_0}{2\pi} = .9869$

$\delta = \frac{\alpha}{\omega_0} = 0.5820$



Damped: $(b > 0)$.

$$mx'' + bx' + kx = 0$$

characteristic equation:

$$r^2 + \frac{b}{m}r + \frac{k}{m} = 0$$

$$r_1, r_2 = \frac{-\frac{b}{m} \pm \sqrt{\left(\frac{b}{m}\right)^2 - 4\frac{k}{m}}}{2}$$

$\frac{k}{m} = \omega_0^2$

Let $\frac{b}{2m} = p$

$$r_1, r_2 = \frac{-b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} = -p \pm \sqrt{p^2 - \omega_0^2}$$

$$\boxed{r_1, r_2 = \frac{-b}{2m} \pm \sqrt{\frac{b^2 - 4mk}{4m^2}}}$$

$$\boxed{r_1, r_2 = \frac{-b}{2m} \pm \sqrt{\frac{b^2 - 4mk}{4m^2}}}$$

Roots could be (1) distinct real

(2) repeated real

(3) complex conjugates.

(1) overdamping

(2) critical damping

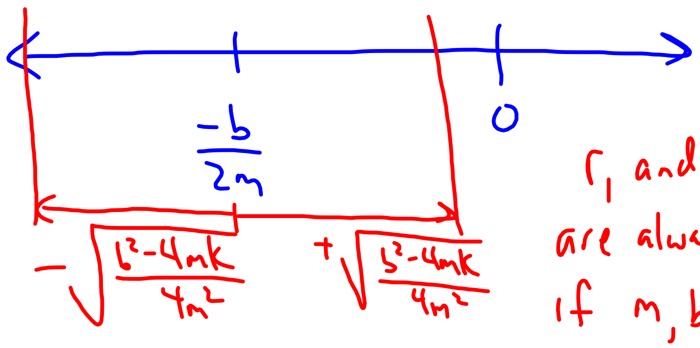
(3) underdamping.

$$\boxed{r_1, r_2 = \frac{-b}{2m} \pm \sqrt{\frac{b^2-4mk}{4m^2}}}$$

If $b^2-4mk > 0$,

- Two distinct real roots
- overdamping.

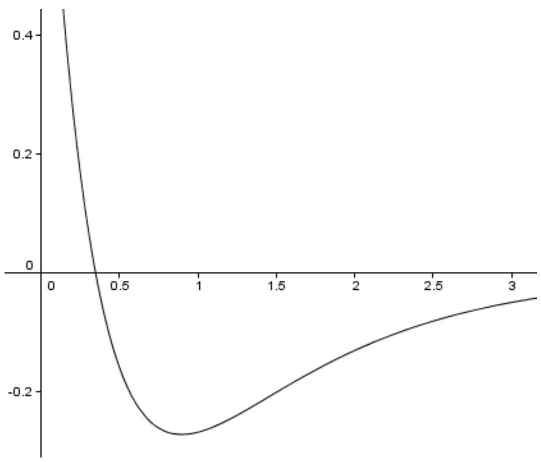
Note that $\sqrt{\frac{b^2-4mk}{4m^2}} < \frac{b}{2m}$.



r_1 and r_2 are always neg. if m, b, k are all positive.

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Since $r_1, r_2 < 0$, these are both exponential decay, so $x(t) \to 0$ as $t \to \infty$.

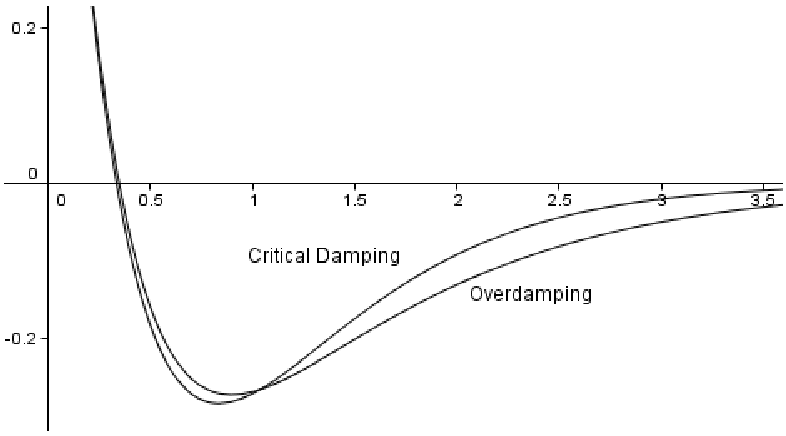


$$\boxed{r_1, r_2 = \frac{-b}{2m} \pm \sqrt{\frac{b^2 - 4mk}{4m^2}}}$$

If $b^2 - 4mk = 0$

- $r_1 = r_2 = \frac{-b}{2m}$ (repeated real roots)
- "Critically damped"

$$x(t) = c_1 e^{\frac{-b}{2m}t} + c_2 \cdot t \cdot e^{\frac{-b}{2m}t}$$



$$\boxed{r_1, r_2 = \frac{-b}{2m} \pm \sqrt{\frac{b^2 - 4mk}{4m^2}}}$$

If $b^2 - 4mk < 0$

- complex conjugate roots with a non-zero a (in $\underline{\underline{a}} + bi$)
- "underdamped"

With $r_1, r_2 = -\frac{b}{2m} \pm \sqrt{\frac{b^2 - 4mk}{4m^2}}$

$$= \alpha \pm \beta i$$

$$x(t) = e^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right)$$

$$= Ce^{\alpha t}\left(\frac{A}{C}\cos\beta t + \frac{B}{C}\sin\beta t\right)$$

$$= Ce^{\alpha t}\left(\cos\left(\beta t - \alpha\right)\right)$$

Here... β is <u>not</u> just $\sqrt{\frac{k}{m}}$.

Solve $x'' + 2x' + 100x = 0$

$x(0) = 1, \quad x'(0) = -5$

(Same as before, but with damping).

$r^2 + 2r + 100 = 0 \rightarrow (r+1)^2 + 99 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 400}}{2} = \boxed{-1 \pm i\sqrt{99}}$$

$$\boxed{x(t) = e^{-t}\left(A\cos(\sqrt{99}\,t) + B\sin(\sqrt{99}\,t)\right)}$$

general solution.

$x(0) = 1 \Longrightarrow A = 1.$

$$x'(t) = -e^{-t}\left(\cos(\sqrt{99}\,t) + B\sin(\sqrt{99}\,t)\right)$$
$$+ e^{-t}\left(-\sqrt{99}\sin(\sqrt{99}\,t) + B\cdot\sqrt{99}\cos(\sqrt{99}\,t)\right)$$

@ $t=0$:

$$-1(1 + 0) + 1(0 + \sqrt{99}\,B) = -5$$

$$-1 + \sqrt{99}\,B = -5$$

$$\boxed{B = \frac{-4}{\sqrt{99}}}$$

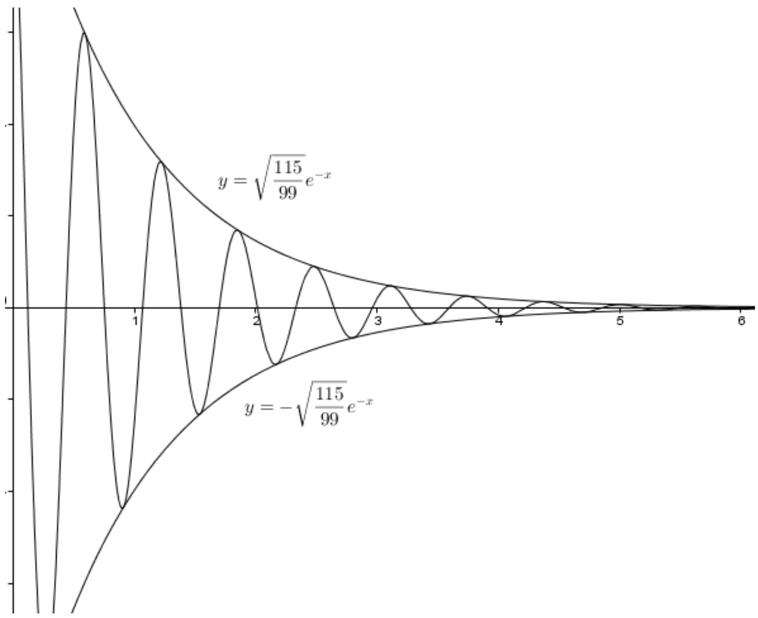
So $\boxed{x(t) = e^{-t}\left(\cos\sqrt{99}\,t - \frac{4}{\sqrt{99}}\sin\sqrt{99}\,t\right)}$

$$C = \sqrt{A^2 + B^2} = \sqrt{1 + \frac{16}{99}} = \sqrt{\frac{115}{99}}$$

$$\alpha = 2\pi - \tan^{-1}\left(\frac{4/\sqrt{99}}{1}\right) = 5.9009$$

$$x(t) = \underbrace{\sqrt{\frac{115}{99}}\, e^{-t}}_{\text{time-varying amplitude}} \cos\left(\sqrt{99}\,t - 5.9009\right)$$

pseudo-frequency


$y = \sqrt{\frac{115}{99}} e^{-x}$
$y = -\sqrt{\frac{115}{99}} e^{-x}$
1
2
3
4
5
6