Solve $2y'' - 7y' + 3y = 0$.

Black box:

Let $y = e^{rt}$.

$y' = re^{rt}$, $y'' = r^2 e^{rt}$

$2r^2e^{rt} - 7re^{rt} + 3e^{rt} = 0$

$e^{rt}(2r^2 - 7r + 3) = 0$

never 0

It's okay to start here as long as you know what happens in the "black box."

$2r^2 - 7r + 3 = 0$

$(2r-1)(r-3) = 0$

$r = \frac{1}{2}$ $\quad r = 3$

$y_1 = e^{\frac{1}{2}t}$ $\quad y_2 = e^{3t}$

$$\boxed{y = c_1 e^{\frac{1}{2}t} + c_2 e^{3t}}$$

Lin Ind?

If characteristic equation leads to distinct roots r_1 & r_2, the sols $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ will be lin. ind. b/c $\dfrac{e^{r_1 t}}{e^{r_2 t}} = e^{(r_1 - r_2)t}$ can never be a constant (since $r_1 \neq r_2$).

Common Check for Ind. of sols to ODE:

Two sols y_1 & y_2 are lin. ind. if

$$c_1 y_1 + c_2 y_2 = 0$$

has only the trivial solution $c_1 = c_2 = 0$.

We need a 2nd eqn!

$$c_1 y_1 + c_2 y_2 = 0$$
$$c_1 y_1' + c_2 y_2' = 0$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This has only the trivial solution if

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

The "Wronskian"

In our example: $y_1 = e^{\frac{1}{2}t}$ $y_2 = e^{3t}$

$$W = \begin{vmatrix} e^{\frac{1}{2}t} & e^{3t} \\ \frac{1}{2}e^{\frac{1}{2}t} & 3e^{3t} \end{vmatrix} = (3e^{3t})(e^{\frac{1}{2}t}) - (e^{3t})(\tfrac{1}{2}e^{\frac{1}{2}t})$$

$$= 3e^{\frac{7}{2}t} - \frac{1}{2}e^{\frac{7}{2}t}$$

$$= \frac{5}{2}e^{\frac{7}{2}t} \neq 0$$

at each value of t.

Now solve

$$y'' + 2y' + y = 0, \quad y(0) = 5, \quad y'(0) = -3$$

$$r^2 + 2r + 1 = 0$$

$$(r+1)(r+1) = 0$$

$$r_1 = -1 \quad r_2 = -1$$

$$y_1 = e^{-t} \quad \boxed{y_2 = e^{-t}}$$

repeated solution, so we do not have two lin. ind. solns that can serve as a basis for what we know should be a 2-dimensional solution space.

Let $y_2 = te^{-t}$.

Are y_1 & y_2 lin. ind?

$$W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix}$$

$$= e^{-2t} - te^{-2t} + te^{-2t} = e^{-2t} \neq 0.$$

So y_1 & y_2 are lin. ind.

$$y(t) = c_1 y_1 + c_2 y_2$$

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

Now solve the IVP:

$$y'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}$$

$$y(0) = c_1 + 0 = 5 \quad \Rightarrow c_1 = 5$$

$$y'(0) = -c_1 + c_2 - 0 = -3 \quad c_2 = 2$$

$$\therefore \boxed{y(t) = 5e^{-t} + 2te^{-t}.}$$