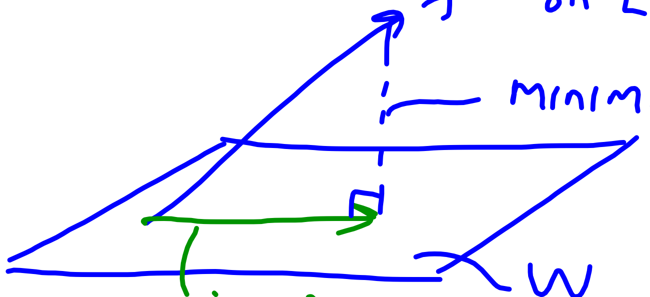
Approximating Functions



f on $[a,b]$

minimal $|e| = |f-g|$.

W spanned by $\{g_1, g_2, \ldots, g_k\}$

$g = \text{proj}_W f$

$f \notin W$

(f is not a linear combination of the g_i)

Difference between f and g is $|f-g|$. Note that

$$|f-g|^2 = \langle f-g, f-g \rangle$$

(minimizing $|f-g|$ is equivalent to minimizing $(f-g)^2$)

$$|f-g|^2 = \langle f-g, f-g \rangle = \int_a^b \big(f(x) - g(x)\big)^2 \, dx$$

(the MSE).

We call g the least-squares approximation to f in W.

Example 1

Find the best linear approximation to

$f(x) = e^x$ on $[-1, 1]$.

Linear $g(x) = a + bx$

Basis: $B = \{1, x\}$

(But are these orthogonal on $[-1,1]$?)

$$\langle 1, x\rangle = \int_{-1}^{1} (1)(x)\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1^2}{2} - \frac{(-1)^2}{2} = 0; \text{ orth.}$$

$$\text{proj}_W(e^x) = \frac{\langle e^x, 1\rangle}{\langle 1, 1\rangle}(1) + \frac{\langle e^x, x\rangle}{\langle x, x\rangle}(x)$$

$$= \frac{\int_{-1}^{1} (e^x)(1)\,dx}{\int_{-1}^{1} (1)(1)\,dx}(1) + \frac{\int_{-1}^{1} xe^x\,dx}{\int_{-1}^{1} (x)(x)\,dx}(x)$$

$$\approx 1.18 + 1.10x.$$

Example 2

Find the best quadratic approximation to $f(x) = e^{x}$ on $[-1, 1]$.

$$g(x) = a + bx + cx^2$$

Basis: $B = \{1, x, x^2\}$.

(we know from the previous example that 1 and x are orthogonal. What about x^2?)

$$\langle 1, x^2\rangle = \int_{-1}^{1} (1)(x^2)\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - \left(\frac{-1}{3}\right) = \frac{2}{3} \neq 0.$$

(not orthogonal).

Use G-S to create an orthogonal basis:

$$V_1 = 1$$

$$V_2 = x$$

$$V_3 = x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle}(1) - \frac{\langle x^2, x\rangle}{\langle x, x\rangle}(x)$$

$$= x^2 - \frac{\int_{-1}^{1}(1)(x^2)\,dx}{\int_{-1}^{1}(1)(1)\,dx}(1) - \frac{\int_{-1}^{1}(x^2)(x)\,dx}{\int_{-1}^{1}(x)(x)\,dx}(x)$$

$$= \boxed{x^2 - \frac{1}{3}}$$

So orth. basis is $\left\{1, x, x^2 - \frac{1}{3}\right\}$.

"Legendre Polynomials"

Also:

Chebyshev $\langle p, g\rangle = \int_{-1}^{1} p(x) g(x) (1-x^2)^{-1/2}\,dx$

Jacobi $\langle p, g\rangle = \int_{-1}^{1} p(x) g(x) (1-x)^{\lambda} (1+x)^{\mu}\,dx$

Hermite

Laguerre

Now project e^x onto W spanned by $\{1, x, x^2 - \frac{1}{3}\}$:

$$e^x \approx \frac{\langle 1, e^x\rangle}{\langle 1,1\rangle}(1) + \frac{\langle x, e^x\rangle}{\langle x,x\rangle}(x) + \frac{\langle x^2-\frac{1}{3}, e^x\rangle}{\langle x^2-\frac{1}{3}, x^2-\frac{1}{3}\rangle}\left(x^2-\frac{1}{3}\right)$$

$$= 1.18 + 1.10x + 0.54x^2 - 0.18$$

$$= 0.54x^2 + 1.10x + \underline{1}.$$

Fourier Approximations

Trigonometric Polynomial:

$$a_0 + a_1\cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$
$$+ a_n \cos(nx) + b_n \sin(nx)$$
$$= a_0 + \sum_{k=1}^{n} \left(a_k \cos(kx) + b_k \sin(kx)\right)$$

A polynomial of the form

$$3 + 2\sin x - 3\cos 2x + 4\sin 2x + \sin 3x$$

is called a "3rd-order trig. polynomial".

- Generally, trig polynomials are developed using an inner product defined on an interval of length 2π, such as $[-\pi,\pi]$ or $[0,2\pi]$.
- Basis: (orthogonal) $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \dots\}$

<u>Coefficients</u> on $[-\pi,\pi]$

$$\langle \sin kx, \sin kx \rangle = \int_{-\pi}^{\pi} (\sin^2 kx)\, dx = \pi$$

$$\langle \cos kx, \cos kx \rangle = \int_{-\pi}^{\pi} (\cos^2 kx)\, dx = \pi$$

$$\langle 1, 1 \rangle = \int_{-\pi}^{\pi} (1)\, dx = 2\pi$$

Also, if $m \neq n$,

$$\langle \sin(mx), \sin(nx) \rangle = 0$$

$$\langle \cos(mx), \cos(nx) \rangle = 0$$

$$\langle \sin(mx), \cos(mx) \rangle = 0$$

$$\langle \sin(mx), \cos(nx) \rangle = 0$$

$$\langle 1, \sin kx \rangle = 0$$

$$\langle 1, \cos kx \rangle = 0$$

2nd-order trig. polynomial for e^x on $[-\pi, \pi]$:

$$e^x \approx a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x.$$

$$a_0 = \frac{\int_{-\pi}^{\pi} (e^x)(1)\, dx}{2\pi} = \frac{\langle 1, e^x \rangle}{\langle 1, 1 \rangle}$$

$$a_1 = \frac{\int_{-\pi}^{\pi} e^x \cos x \, dx}{\pi} = \frac{\langle e^x, \cos x \rangle}{\langle \cos x, \cos x \rangle}$$

$$b_1 = \frac{\int_{-\pi}^{\pi} e^x \sin x \, dx}{\pi} = \frac{\langle e^x, \sin x \rangle}{\langle \sin x, \sin x \rangle}$$

$$a_2 = \frac{\int_{-\pi}^{\pi} e^x \cos(2x) \, dx}{\pi} = \frac{\langle e^x, \cos 2x \rangle}{\langle \cos 2x, \cos 2x \rangle}$$

$$b_2 = \frac{\int_{-\pi}^{\pi} e^x \sin(2x) \, dx}{\pi} = \frac{\langle e^x, \sin 2x \rangle}{\langle \sin 2x, \sin 2x \rangle}$$

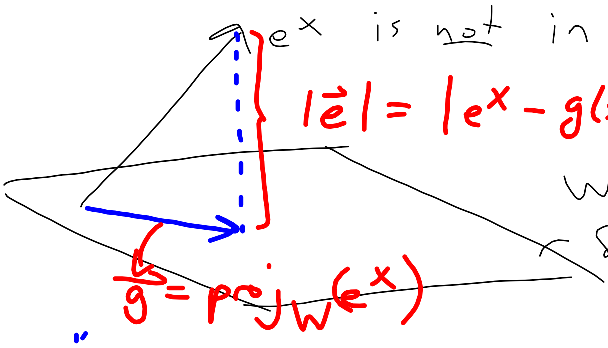
We'd like to approximate a function with some other set of functions...

e^x is not of the form $a+bx$, i.e., e^x cannot be written as a linear combination of 1 and x on any interval, specifically on $[-1, 1]$.

e^x is not in the subspace spanned by 1 & x.

Linear functions like $a+bx$ have $\{1, x\}$ as a basis.

e^x is <u>not</u> in W.



$W =$ set of all $a+bx$ with basis $B = \{1, x\}$.

"Best" linear approximation to e^x will be $g(x) = \text{proj}_W(e^x)$.

* "Best" implies $|e^x - g(x)|$ is minimized.

Instead of minimizing $|e^x - g(x)|$, we typically minimize the squared length, i.e.

$$|e^x - g(x)|^2 = \langle f-g, f-g \rangle$$

(where $f(x) = e^x$).

$$= \int_a^b [f(x)-g(x)]^2\, dx$$

$= MSE$ (mean-squared error).

(The orthogonal projection of f onto W minimizes this MSE.)

We want to project e^x onto W, where W has a basis of $\{1, x\}$ for $[-1,1]$.

Are these orthogonal on $[-1,1]$?

Yes, if $\langle 1, x \rangle = 0$.

$$\int_{-1}^{1} (1)(x)\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1}{2} - \left(\frac{1}{2}\right) = 0.$$

$$\therefore \ \text{proj}_W(e^x) = \frac{\langle f, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle f, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$= \frac{\int_{-1}^{1} (e^x)(1)\,dx}{\int_{-1}^{1} (1)(1)\,dx}(1) + \frac{\int_{-1}^{1} (e^x)(x)\,dx}{\int_{-1}^{1} (x)(x)\,dx}(x)$$

$$= 1.18 + 1.10x$$

Now find the best quadratic approximation to e^x on the interval $[-1,1]$.

$$e^x \approx a + bx + cx^2$$

Basis: $\{1, x, x^2\}$.

(Are the basis vectors still orthogonal?)

$\langle 1, x\rangle = 0$ (from previous example on $[-1,1]$)

$\langle 1, x^2\rangle = \int_{-1}^{1}(1)(x^2)\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - \frac{(-1)}{3} = \frac{2}{3} \neq 0.$

(not an orthogonal basis)

G-S to the rescue!

$v_1 = 1$

$v_2 = x$

$$v_3 = x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle}(1) - \frac{\langle x^2, x\rangle}{\langle x, x\rangle}(x)$$

$$= x^2 - \frac{\int_{-1}^{1}(x^2)(1)\,dx}{\int_{-1}^{1}(1)(1)\,dx}(1) - \frac{\int_{-1}^{1}(x^2)(x)\,dx}{\int_{-1}^{1}(x)(x)\,dx}(x)$$

$$= x^2 - \frac{1}{3}.$$

$$B = \left\{1, x, x^2 - \frac{1}{3}\right\}$$

Finally, $e^x \approx a + bx + cx^2$ ...

$$e^x \approx \frac{\int_{-1}^{1}(e^x)(1)\,dx}{\int_{-1}^{1}(1)(1)\,dx}(1) + \frac{\int_{-1}^{1}(e^x)(x)\,dx}{\int_{-1}^{1}(x)(x)\,dx}(x) + \frac{\int_{-1}^{1}(e^x)(x^2-\frac{1}{3})\,dx}{\int_{-1}^{1}(x^2-\frac{1}{3})(x^2-\frac{1}{3})\,dx}\left(x^2-\frac{1}{3}\right)$$

$$= 0.54x^2 + 1.1x + 1$$

"Fourier Series"

⇒ Four. Poly → Four Trans → DFT
→ FFT

Trigonometric Polynomial of <u>nth</u> order:

If a_n & b_n are not <u>both</u> zero

$$a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos(nx)$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin(nx)$$
$$= a_0 + \sum_{k=1}^{n} \left[a_k \cos kx + b_k \sin kx \right]$$

Typically we calculate Trig. Poly. on intervals of length 2π, such as $[-\pi, \pi]$ or $[0, 2\pi]$.

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx .$$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx .$$

Conveniently:

The basis $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots\}$ is an orthogonal basis on $[-\pi, \pi]$.

- $\langle 1, 1 \rangle = \int_{-\pi}^{\pi} (1)(1)\, dx = 2\pi .$
- $\langle \sin kx, \sin kx \rangle = \int_{-\pi}^{\pi} (\sin^2 kx)\, dx = \pi .$
- $\langle \cos kx, \cos kx \rangle = \int_{-\pi}^{\pi} (\cos^2 kx)\, dx = \pi .$

Find a 2nd-order Fourier approx. to $y = e^x$ on $[-\pi, \pi]$.

$$B = \{1, \cos x, \sin x, \cos 2x, \sin 2x\}$$

$$a_0 = \frac{\langle e^x, 1\rangle}{\langle 1, 1\rangle} = \frac{\int_{-\pi}^{\pi} e^x\,dx}{2\pi} = 3.676$$

$$a_1 = \frac{\langle e^x, \cos x\rangle}{\pi} = \frac{\int_{-\pi}^{\pi} e^x \cos x\,dx}{\pi} = -3.676$$

$$a_2 = \frac{\langle e^x, \cos 2x\rangle}{\pi} = \frac{\int_{-\pi}^{\pi} e^x \cos 2x\,dx}{\pi} = 1.470$$

$$b_1 = \frac{\langle e^x, \sin x\rangle}{\pi} = \frac{\int_{-\pi}^{\pi} e^x \sin x\,dx}{\pi} = 3.676$$

$$b_2 = \frac{\langle e^x, \sin 2x\rangle}{\pi} = \frac{\int_{-\pi}^{\pi} e^x \sin 2x\,dx}{\pi} = -2.941$$

$$e^x \approx 3.676 - 3.676\cos x + 1.470\cos 2x + 3.676\sin x - 2.941\sin 2x$$