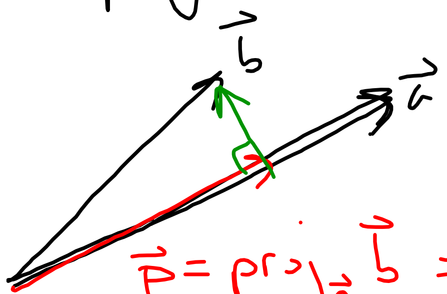
"Gram-Schmidt Orthogonalization"

We can project one vector onto another:



$$\vec p = \text{proj}_{\vec a}\vec b = c\vec a = \left(\frac{\vec a\cdot\vec b}{\vec a\cdot\vec a}\right)\vec a$$

$\vec a\cdot\vec a$

$= a^T a \to$ 1-by-1

$a a^T \to n\times n$

(or via projection matrix

$$P = \frac{\vec a\vec a^T}{\vec a^T\vec a}$$

$$\Rightarrow \vec p = P\vec b\ )$$

We can also project onto a subspace.

Given an orthogonal basis

$B = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ for the

subspace $\underline{V}$, the projection of a

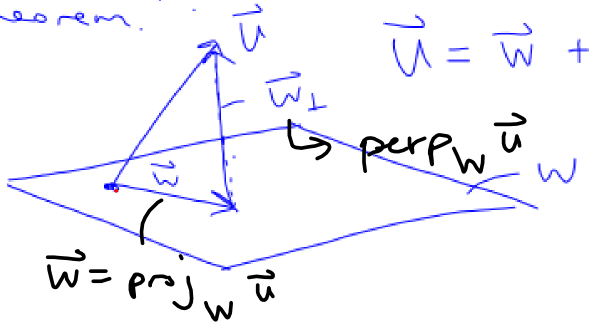
vector $\vec{v}$ onto the subspace is given

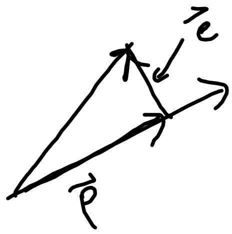
by

$$\text{proj}_{\underline{V}} \vec{v} = \frac{\vec{v}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{v}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 + \cdots + \frac{\vec{v}\cdot\vec{u}_k}{\vec{u}_k\cdot\vec{u}_k}\vec{u}_k .$$

This leads to the "orthogonal projection Theorem.":

$\vec{u} = \vec{w} + \vec{w}_{\perp}$





$\vec{p} + \vec{e} = \vec{b}$

$\vec{u} = \vec{w} + \vec{w}_{\perp}$ (where $\vec{w} \in W$)

$\vec{u} = \left(\sum \mathrm{proj}_{w_i} \vec{u}\right) + \vec{w}_{\perp}$

$\Rightarrow$

$\vec{w}_{\perp} = \vec{u} - \sum \mathrm{proj}_{\vec{w}_i} \vec{u}$

Find the projection of $(1,2,3,4)$ onto the subspace of $\mathbb{R}^4$ spanned by $\{(1,2,0,3),(4,0,5,8),(8,1,5,6)\}$.

These are not orthogonal basis vectors

Can we construct orthogonal basis vectors from these?

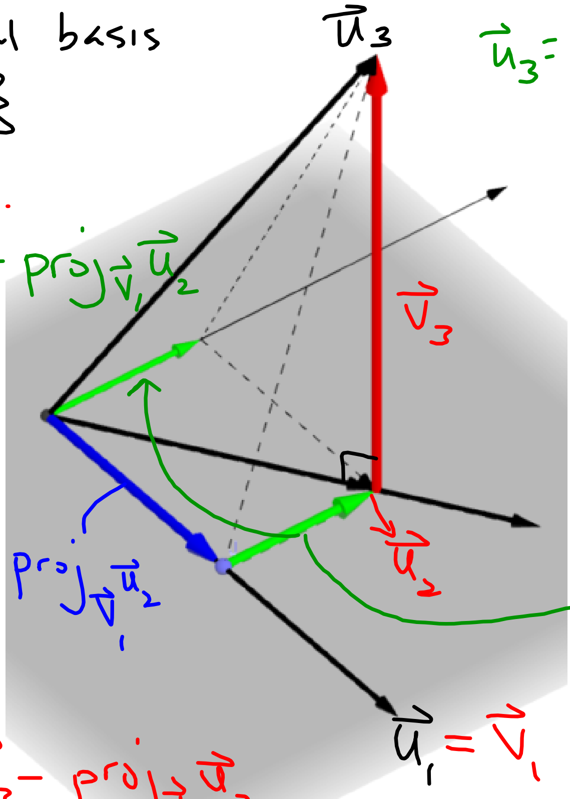
- not-orthogonal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$

Let $\vec{v}_1 = \vec{u}_1$.

Then $\vec{v}_2 = \vec{u}_2 - \text{proj}_{\vec{v}_1}\vec{u}_2$

And

$\vec{u}_3 = \text{proj}_{v_1} u_3 + \text{proj}_{v_2} u_3 + v_3$



$\text{proj}_{\vec{v}_1}\vec{u}_2$

$\vec{u}_2 - \text{proj}_{\vec{v}_1}\vec{u}_2$

$\vec{v}_3 = \vec{u}_3 - \text{proj}_{\vec{v}_1}\vec{u}_3 - \text{proj}_{\vec{v}_2}\vec{u}_3$

Then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ $\underline{\underline{\text{is}}}$ orthogonal!

(And $\left\{ \dfrac{\vec{v}_1}{|\vec{v}_1|}, \dfrac{\vec{v}_2}{|\vec{v}_2|}, \dfrac{\vec{v}_3}{|\vec{v}_3|} \right\}$ is orthonormal.)

$$B = \{ (1,2,0,3), (4,0,5,8), (8,1,5,6) \}$$

Gram-Schmidt:

Let $\vec{v}_1 = (1,2,0,3)$

Then $\vec{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 5 \\ 8 \end{bmatrix} - \dfrac{\begin{bmatrix} 4 \\ 0 \\ 5 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$

$$= \begin{bmatrix} 4 \\ 0 \\ 5 \\ 8 \end{bmatrix} - \frac{28}{14} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix}$$

And $\vec{v}_3 = \begin{bmatrix} 8 \\ 1 \\ 5 \\ 6 \end{bmatrix} - \dfrac{\begin{bmatrix} 8 \\ 1 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}}{14} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} - \dfrac{\begin{bmatrix} 8 \\ 1 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix}}{\begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix}} \begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix}$

did in previous step!

$$= \begin{bmatrix} 8 \\ 1 \\ 5 \\ 6 \end{bmatrix} - \frac{28}{14} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} - \frac{49}{49} \begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix}$$

So our orthogonal basis is:

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix} \right\}.$$

Orthonormalize:

$$B_3 = \left\{ \frac{(1,2,0,3)}{\sqrt{14}}, \frac{(2,-4,5,2)}{7}, \frac{(4,1,0,-2)}{\sqrt{21}} \right\}$$