$$A = \begin{bmatrix} 9 & -2 & -1 \\ -6 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix} \cdot \frac{1}{9} \qquad E_1 = \begin{bmatrix} 1/9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to L = \begin{bmatrix} 9 & 0 & 0 \\ & & 0 \\ & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2/9 & -1/9 \\ -6 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix} + 6R_1 \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to L = \begin{bmatrix} 9 & 0 & 0 \\ -6 & & 0 \\ & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2/9 & -1/9 \\ 0 & -1/3 & 1/3 \\ 2 & -1 & 0 \end{bmatrix} -2R_1 \qquad \cdots\cdots \qquad \begin{bmatrix} 9 & 0 & 0 \\ -6 & & 0 \\ 2 & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2/9 & -1/9 \\ 0 & -1/3 & 1/3 \\ 0 & -5/9 & 2/9 \end{bmatrix} \cdot -3 \qquad \begin{bmatrix} 9 & 0 & 0 \\ -6 & -1/3 & 0 \\ 2 & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2/9 & -1/9 \\ 0 & 1 & -1 \\ 0 & -5/9 & 2/9 \end{bmatrix} + \frac{5}{9}R_2 \qquad \begin{bmatrix} 9 & 0 & 0 \\ -6 & -1/3 & 0 \\ 2 & -5/9 & \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2/9 & -1/9 \\ 0 & 1 & -1 \\ 0 & 0 & -1/3 \end{bmatrix} \cdot (-3) \qquad \underbrace{\begin{bmatrix} 9 & 0 & 0 \\ -6 & -1/3 & 0 \\ 2 & -5/9 & -1/3 \end{bmatrix}}_{L}$$

$$\underbrace{\begin{bmatrix} 1 & -2/9 & -1/9 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$A = LU.$

$$A = \begin{bmatrix} 9 & -2 & -1 \\ -6 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ -6 & -1/3 & 0 \\ 2 & -5/9 & -1/3 \end{bmatrix}\begin{bmatrix} 1 & -2/9 & -1/9 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve $A\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. Since $A = LU$,

$$* \quad L\underbrace{U\vec{x}}_{3\times1} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$L\vec{y} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 9 & 0 & 0 \\ -6 & -1/3 & 0 \\ 2 & -5/9 & -1/3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$9y_1 = 2 \longrightarrow \boxed{y_1 = 2/9}$$

$$-6y_1 - \tfrac{1}{3}y_2 = -1$$

$$-\tfrac{1}{3}y_2 = -1 + \tfrac{12}{9}$$

$$\boxed{y_2 = 3 - 4 = -1}$$

$$2\left(\tfrac{2}{9}\right) - \tfrac{5}{9}(-1) - \tfrac{1}{3}y_3 = 0$$

$$\boxed{y_3 = 3}$$

And since $\vec{y} = U\vec{x}$, we now have

$$\begin{bmatrix} 1 & -2/9 & -1/9 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/9 \\ -1 \\ 3 \end{bmatrix}$$

$$x_3 = 3$$

$$x_2 - x_3 = x_2 - 3 = -1 \Rightarrow x_2 = 2$$

$$x_1 - \tfrac{2}{9}(2) - \tfrac{1}{9}(3) = 2/9 \qquad \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x_1 = \tfrac{2}{9} + \tfrac{4}{9} + \tfrac{3}{9} = 1$$

Why do we like orthogonal (or orthonormal) bases so much?

Let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be an orthogonal basis for a vector space V.

(orthogonality $\Longrightarrow \vec{u}_i \cdot \vec{u}_j = 0$ if $i \neq j$)

Since the $\vec{u}_i$ form a basis for V, any vector $\vec{v} \in V$ can be written as a linear combination

$$\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$$

How can we find the scalars c_i?

We've typically done this by solving a system of equations. But in this case we can use orthogonality of $\vec{u}_i$ to help!

Take dot products of $\vec{v}$ with each $\vec{u}_i$:

$$\vec{v}\cdot\vec{u}_1 = c_1\vec{u}_1\cdot\vec{u}_1 + \underbrace{\cancel{c_2\vec{u}_2\cdot\vec{u}_1}}_{0} + \cdots + \underbrace{\cancel{c_n\vec{u}_n\cdot\vec{u}_1}}_{0}$$

$$\Longrightarrow \boxed{c_1 = \frac{\vec{v}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}}$$

$$\vec{v}\cdot\vec{u}_2 = \underbrace{\cancel{c_1\vec{u}_1\cdot\vec{u}_2}}_{0} + c_2\vec{u}_2\cdot\vec{u}_2 + \cdots + \underbrace{\cancel{c_n\vec{u}_n\cdot\vec{u}_2}}_{0}$$

$$\Longrightarrow \boxed{c_2 = \frac{\vec{v}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}}$$

$\vdots$

$$\Longrightarrow \boxed{c_i = \frac{\vec{v}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}}$$

So... the scalars in

$$\vec{v} = \underbrace{c_1}\vec{u}_1 + \underbrace{c_2}\vec{u}_2 + \cdots + \underbrace{c_n}\vec{u}_n$$

are

$$\vec{v} = \underbrace{\frac{\vec{u}_1 \cdot \vec{v}}{\vec{u}_1 \cdot \vec{u}_1}}\vec{u}_1 + \underbrace{\frac{\vec{u}_2 \cdot \vec{v}}{\vec{u}_2 \cdot \vec{u}_2}}\vec{u}_2 + \cdots + \underbrace{\frac{\vec{u}_n \cdot \vec{v}}{\vec{u}_n \cdot \vec{u}_n}}\vec{u}_n$$

Bonus! If the basis is an orthonormal basis, the length of each basis vector is 1. Then $\vec{u}_i \cdot \vec{u}_i = 1$. Then:

$$\vec{v} = \underbrace{(\vec{u}_1 \cdot \vec{v})}_{c_1}\vec{u}_1 + \underbrace{(\vec{u}_2 \cdot \vec{v})}_{c_2}\vec{u}_2 + \cdots + \underbrace{(\vec{u}_n \cdot \vec{v})}_{c_n}\vec{u}_n$$

So... suppose we use

$$B_1 = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \{(0,-1,0), (3,0,-4), (4,0,3)\}$$

as an orthogonal basis for $\mathbb{R}^3$.

Write $\vec{v} = (2, 3, -1)$ as a lin. comb. of the vectors in B.

OLD WAY: $c_1\begin{bmatrix}0\\-1\\0\end{bmatrix} + c_2\begin{bmatrix}3\\0\\-4\end{bmatrix} + c_3\begin{bmatrix}4\\0\\3\end{bmatrix} = \begin{bmatrix}2\\3\\-1\end{bmatrix}$?

NEW WAY: $c_1 = \dfrac{(0,-1,0)\cdot(2,3,-1)}{(0,-1,0)\cdot(0,-1,0)} = \dfrac{-3}{1} = -3.$

$$c_2 = \frac{(3,0,-4)\cdot(2,3,-1)}{(3,0,-4)\cdot(3,0,-4)} = \frac{10}{25} = \frac{2}{5}$$

$$c_3 = \frac{(4,0,3)\cdot(2,3,-1)}{(3,0,-4)\cdot(3,0,-4)} = \frac{5}{25} = \frac{1}{5}$$

$$\therefore \begin{bmatrix}2\\3\\-1\end{bmatrix} = -3\begin{bmatrix}0\\-1\\0\end{bmatrix} + \frac{2}{5}\begin{bmatrix}3\\0\\-4\end{bmatrix} + \frac{1}{5}\begin{bmatrix}4\\0\\3\end{bmatrix}.$$

BONUS NEW WAY

Now use the orthonormal basis:

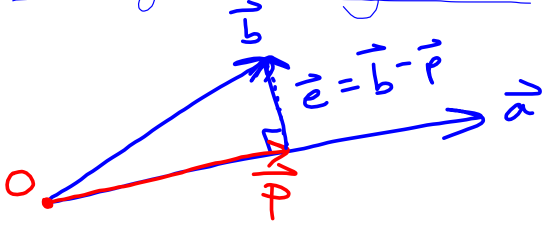
$$\left\{(0,-1,0), \left(\tfrac{3}{5}, 0, -\tfrac{4}{5}\right), \left(\tfrac{4}{5}, 0, \tfrac{3}{5}\right)\right\}$$

$$\begin{bmatrix}2\\3\\-1\end{bmatrix} = \left(\begin{bmatrix}0\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}2\\3\\-1\end{bmatrix}\right)\begin{bmatrix}0\\-1\\0\end{bmatrix} + \left(\begin{bmatrix}3/5\\0\\-4/5\end{bmatrix}\cdot\begin{bmatrix}2\\3\\-1\end{bmatrix}\right)\begin{bmatrix}3/5\\0\\-4/5\end{bmatrix} + \left(\begin{bmatrix}4/5\\0\\3/5\end{bmatrix}\cdot\begin{bmatrix}2\\3\\-1\end{bmatrix}\right)\begin{bmatrix}4/5\\0\\3/5\end{bmatrix}$$

$$\vec{v} = (\vec{u}_1\cdot\vec{v})\vec{u}_1 + (\vec{u}_2\cdot\vec{v})\vec{u}_2 + (\vec{u}_3\cdot\vec{v})\vec{u}_3$$

$$= -3\begin{bmatrix}0\\-1\\0\end{bmatrix} + \frac{10}{5}\begin{bmatrix}3/5\\0\\-4/5\end{bmatrix} + \frac{5}{5}\begin{bmatrix}4/5\\0\\3/5\end{bmatrix}$$

Orthogonal Projections



$\vec{p} = \text{proj}_{\vec{a}} \vec{b}$

(the projection of $\vec{b}$ onto $\vec{a}$)

$\vec{e} = \vec{b} - \vec{p}$ $\qquad \vec{p} = c\vec{a}$

We know $\vec{a}$ & $\vec{b}$. We want $\vec{p}$. Since $\vec{p} = c\vec{a}$, what is c?

$\vec{e} \perp \vec{p}$

$\vec{e} \perp c\vec{a}$

$\vec{e} \perp \vec{a}$

$\vec{b} - \vec{p} \perp c\vec{a}$

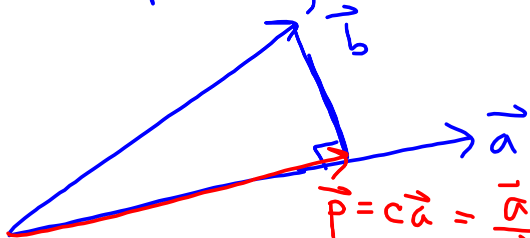
$$(\vec{b} - \vec{p}) \cdot (\vec{a}) = 0$$

$$(\vec{b} - c\vec{a}) \cdot (\vec{a}) = 0$$

$$\vec{b} \cdot \vec{a} - c\vec{a} \cdot \vec{a} = 0$$

$$\boxed{c = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}}$$

$\therefore$ The projection of $\vec{b}$ onto $\vec{a}$ is $\vec{p} = c\vec{a}$, where $c = \dfrac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}$.



$\vec{p} = c\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a}$

$$\boxed{\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a}}$$

$$\boxed{\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a}}$$

$$\left\{\begin{array}{l} \vec{a}\cdot\vec{b} = \vec{a}^T\vec{b} \\ \vec{a}\cdot\vec{a} = \vec{a}^T\vec{a} \end{array}\right.$$

$$\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a} = \vec{a}\,\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}$$

$$= \vec{a}\,\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}$$

$\vec{a} \to n\times 1$
$\vec{a}^T \to 1\times n$
$\vec{b} \to n\times 1$

$$= \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}\vec{b}$$

$n\times n$ matrix

Scalar

$$\underbrace{\frac{\boxed{\vec{a}\vec{a}^T}}{\boxed{\vec{a}^T\vec{a}}}}_{P}\ \boxed{\vec{b}} = \vec{x}$$

$n\times 1$ vector

$(n\times 1)$

P "projection matrix"

Two approaches to projection of $\vec{b}$ onto $\vec{a}$:

$$\vec{p} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a} \qquad (\text{scalar multiple of } \vec{a})$$

$$\vec{p} = P\vec{b} \qquad (\text{projection matrix times } \vec{b})$$

$$= \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}\vec{b}$$

project $\underset{\vec{b}}{(6,7)}$ onto $\underset{\vec{a}}{(1,4)}$.

$$\text{proj}_{\vec{a}}\vec{b} = \frac{(6,7)\cdot(1,4)}{(1,4)\cdot(1,4)}(1,4)$$

$$= \frac{34}{17}(1,4)$$

$$= (2,8).$$