Example 2 *(EASY) Find the coordinate matrix of* $\mathbf{x} = (-2, 1, 3)$ *in* R^3 *relative to the standard basis* $\mathbf{S} = \{(1,0,0), (0,1,0), (0,0,1)\}$.

$$[\vec{x}]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

(means $\vec{x} = -2\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\end{pmatrix}$.)

Example 3 *(EASY) The coordinate matrix of* $\mathbf{x}$ *in* R^2 *relative to the ordered basis* $\mathbf{B} = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1,0), (1,2)\}$ *is*

$$[\mathbf{x}]_\mathbf{B} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Find the coordinates of $\mathbf{x}$ *relative to the standard basis* $\mathbf{S} = \{(1,0), (0,1)\}$.

$$[\vec{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Longrightarrow \vec{x} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

$$[\vec{x}]_S = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Note: $$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{P_{B\to S}} \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{[\vec{x}]_B} = \underbrace{\begin{bmatrix} 5 \\ 4 \end{bmatrix}}_{[\vec{x}]_S}$$

$[\vec{x}]_S$ ↳ coordinate vector of $\vec{x}$ relative to standard basis

$[\vec{x}]_B$ → coordinate vector of x relative to basis B

$P_{B\to S}$ ↳ "change-of-basis matrix" or "transition matrix" from B to S.

Also: $$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Implies $$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$[\vec{x}]_B = P^{-1}_{B\to S}\,[\vec{x}]_S$$

i.e., $$\boxed{P^{-1}_{B\to S} = P_{S\to B}.}$$

Example 4 *(TOUGHER) Find the coordinate matrix of* $\mathbf{x} = (1, 2, -1)$ *in* R^3 *relative to the basis*

$$\mathbf{B} = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}.$$

Find scalars c_1, c_2, c_3 such that

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$P_{B \to S} \quad [\vec{x}]_B = [\vec{x}]_S$$

$$\Longrightarrow [\vec{x}]_B = \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}^{-1}}_{P_{S \to B}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

$P_{S \to B}$. — change-of-basis matrix

(or rref $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 2 \\ 1 & 2 & -5 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{array}\right]$)

<u>Note:</u> I could ask for either the change-of-basis matrix <u>or</u> the coordinate vector, or both.

Given $B_1 = \{\vec{u}_1, \vec{u}_2\} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \end{bmatrix} \right\}$

$B_2 = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$,

Determine the coordinate vector of $\begin{bmatrix} -1 \\ 4 \end{bmatrix}_{B_1}$ relative to B_2.

We know how to go from $S \to B$ & from $B \to S$. So we could take this approach:

$$[\vec{x}]_S = -1\vec{u}_1 + 4\vec{u}_2 = -1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 25 \end{bmatrix}$$

$$[\vec{x}]_{B_2} = \underbrace{\underbrace{\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}}_{P_{B_2 \to S}}{}^{-1}}_{P^{-1}_{B_2 \to S} = P_{S \to B_2}} \begin{bmatrix} 7 \\ 25 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} 7 \\ 25 \end{bmatrix} = \begin{bmatrix} 29 \\ -40 \end{bmatrix}.$$

Again, to go from B_1 to $B_2 = \{\vec{v}_1, \vec{v}_2\}$

$$\begin{bmatrix} -1 \\ 4 \end{bmatrix}_{B_1} = -1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

$$= -1(a_1\vec{v}_1 + a_2\vec{v}_2) + 4(b_1\vec{v}_1 + b_2\vec{v}_2)$$

$$= \alpha\vec{v}_1 + \beta\vec{v}_2$$

$$\Rightarrow \begin{bmatrix} -1 \\ 4 \end{bmatrix}_{B_1} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{B_2}$$

We need to know how to write $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ in terms of $\vec{v}_1$ & $\vec{v}_2$ from B_2.

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 5 & 3 & 3 \end{array}\right] \cdots \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & 2 & 2 \\ 5 & 3 & 7 \end{array}\right] \cdots \to \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & -11 \end{array}\right]$$

collapse into: $\left[\begin{array}{cc|cc} 3 & 2 & 1 & 2 \\ 5 & 3 & 3 & 7 \end{array}\right]$.

vecs of B_2 as cols — vecs of B_1 as cols

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 3 & 8 \\ 0 & 1 & -4 & -11 \end{array}\right]$$

So... $\begin{bmatrix} -1 \\ 4 \end{bmatrix}_{B_1} = -1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 7 \end{bmatrix}$

$$= -1\left(3\begin{bmatrix} 3 \\ 5 \end{bmatrix} + (-4)\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) + 4\left(8\begin{bmatrix} 3 \\ 5 \end{bmatrix} + (-11)\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$$

$$= (29)\begin{bmatrix} 3 \\ 5 \end{bmatrix} + (-40)\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow [\vec{x}]_{B_2} = \begin{bmatrix} 29 \\ -40 \end{bmatrix}.$$

Short version:

$$\left[\begin{array}{cc|cc} 3 & 2 & 1 & 2 \\ 5 & 3 & 3 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 3 & 8 \\ 0 & 1 & -4 & -11 \end{array}\right]$$

vecs of B_2 as cols — vecs of B_1 as cols — I — $P_{B_1 \to B_2}$

Therefore $[\vec{x}]_{B_2} = \begin{bmatrix} 3 & 8 \\ -4 & -11 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix}_{B_1}$

$$[\vec{x}]_{B_2} = \begin{bmatrix} 29 \\ -40 \end{bmatrix}$$

Non-standard basis to another non-standard basis

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \end{bmatrix} \right\}, \quad B_2 = \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}.$$

Express $\begin{bmatrix} -1 \\ 4 \end{bmatrix}_{B_1}$ as a coordinate vector relative to B_2.

We know
$$\left.\begin{aligned} [\vec{x}]_{B_1} &= P_{B_1 \to S} [\vec{x}]_S. \\ [\vec{x}]_{B_2} &= P_{B_2 \to S} [\vec{x}]_S. \end{aligned}\right\}$$

Then $[\vec{x}]_{B_2} = P_{B_2 \to S} \left(P^{-1}_{B_1 \to S} [\vec{x}]_{B_1} \right)$

Therefore $[\vec{x}]_{B_2} = \begin{bmatrix} 3 & 8 \\ -4 & -11 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix}_{B_1}$

$$[\vec{x}]_{B_2} = \begin{bmatrix} 29 \\ -40 \end{bmatrix}$$

Example 3 *(EASY)* *The coordinate matrix of* $\mathbf{x}$ *in* R^2 *relative to the ordered basis* $\mathbf{B} = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1,0),(1,2)\}$ *is*

$$[\mathbf{x}]_{\mathbf{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Find the coordinates of $\mathbf{x}$ *relative to the standard basis* $\mathbf{S} = \{(1,0),(0,1)\}$.

From $B \to S$.

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right] \to \text{(already reduced)}$$

So $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is the change-of-basis matrix.

i.e., $[\vec{x}]_S = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

Let $B_1 = \left\{ \begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix} \right\}$

$B_2 = \left\{ \begin{bmatrix} -6 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} \right\}$

(A) Find a transition matrix from B_1 to B_2. $P_{B_1 \to B_2} = \begin{bmatrix} .75 & .75 & 1/12 \\ -.75 & -17/12 & -17/12 \\ 0 & 2/3 & 2/3 \end{bmatrix}$

(B) Find $[\vec{x}]_{B_1}$ if $\vec{x} = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$. $P_{S \to B_1} \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(C) Use (A) & (B) to find $[\vec{x}]_{B_2}$.

$[\vec{x}]_{B_2} = P_{B_1 \to B_2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 19/12 \\ -43/12 \\ 16/12 \end{bmatrix}$

(D) Find $P_{B_2 \to B_1}$ in two ways:

(i) Like you did for $P_{B_1 \to B_2}$ in (A)

(ii) As $P_{B_1 \to B_2}^{-1}$ (from A)

$$\begin{bmatrix} 0 & -4/3 & -17/6 \\ 3/2 & 3/2 & 3 \\ -3/2 & -3/2 & -3/2 \end{bmatrix}$$