RECAP

Dot product (a.k.a. scalar product):

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \vec{u}^T\vec{v}$$

"Length" of $\vec{v}$: (Euclidean Norm in $\mathbb{R}^n$)

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{v_1v_1 + v_2v_2 + \cdots + v_nv_n} = \sqrt{\vec{v}\cdot\vec{v}}$$

$$\Longleftrightarrow$$

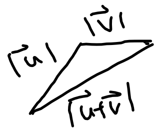
$$|\vec{v}|^2 = \vec{v}\cdot\vec{v}$$

CAUCHY-SCHWARZ Inequality

$$|\vec{u}\cdot\vec{v}| \leq |\vec{u}|\,|\vec{v}|$$

TRIANGLE Inequality

$$|\vec{u}+\vec{v}| \leq |\vec{u}| + |\vec{v}|$$



Angle between vectors $\vec{u}$ & $\vec{v}$:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}$$

If $\theta = 90°$, $\cos\theta = 0 \Longleftrightarrow \vec{u}\cdot\vec{v} = 0.$

Vectors $\vec{u}$ & $\vec{v}$ are orthogonal if $\vec{u}\cdot\vec{v} = 0$.

Any set of mutually orthogonal vectors will linearly independent.

ORTHOGONAL COMPLEMENT

Consider $A\vec{x} = \vec{0}$:

$$\begin{bmatrix} — & \vec{r}_1 & — \\ — & \vec{r}_2 & — \\ & \vdots & \\ — & \vec{r}_m & — \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

If $\vec{x}$ is <u>any</u> vector that solves this system ($A\vec{x} = \vec{0}$), it must be true that

$$\vec{r}_1 \cdot \vec{x} = 0$$
$$\vec{r}_2 \cdot \vec{x} = 0$$
$$\vdots$$
$$\vec{r}_m \cdot \vec{x} = 0$$

So $\vec{x}$ is orthogonal to <u>every</u> row of A.

Therefore, $\vec{x}$ is orthogonal to every linear combination of the rows of A. Why?

Let $\vec{u} = c_1\vec{r}_1 + c_2\vec{r}_2 + \cdots + c_m r_m$ be <u>any</u> linear combination of the rows of A.

Then
$$\begin{aligned} \vec{u} \cdot \vec{x} &= (c_1\vec{r}_1 + c_2\vec{r}_2 + \cdots + c_m\vec{r}_m) \cdot \vec{x} \\ &= c_1\vec{r}_1 \cdot \vec{x} + c_2\vec{r}_2 \cdot \vec{x} + \cdots + c_m\vec{r}_m \cdot \vec{x} \\ &= 0 + 0 + \cdots + 0 \\ &= 0. \end{aligned}$$

Since every $\vec{x}$ in null(A) is orthogonal to every $\vec{u}$ in row(A), we call null(A) and row(A) "orthogonal complements" of one another.

Formally:

The vector $\vec{u}$ is orthogonal to the subspace V of $\mathbb{R}^n$ provided that $\vec{u}$ is orthogonal to every vector in V. The orthogonal complement $V^{\perp}$ of V is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to V.

Let V be a subspace of $\mathbb{R}^n$. Then:

(1) $V^\perp$ is also a subspace of $\mathbb{R}^n$.

(2) The only vector in V and $V^\perp$ is $\vec{0}$.

(3) The ortogonal complement of $V^\perp$ is V.
(i.e., $(V^\perp)^\perp = V$.

Why "complements"?

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\dim(\text{null}(A)) = \text{nullity}(A) = \#$ of free vars $= 1$.

$\dim(\text{row}(A)) = \#$ of non-zero rows in reduced ref.

null(A) — 1-dim subspace of $\mathbb{R}^3$.

row(A) — 2-dim subspace of $\mathbb{R}^3$.

$\boxed{\text{null}(A) \perp \text{row}(A)}$

since $\text{col}(A) = \text{row}(A^T)$,

$\boxed{\text{null}(A^T) \perp \text{col}(A).}$

General Vector Spaces

Let $M_{\mathbb{C}}$ be the set of all 2x2 matrices of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \overset{Re(z)}{a} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \overset{Im(z)}{b} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

basis for M_c.

Is M_c a subspace?

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix} + \begin{bmatrix} e & -f \\ f & e \end{bmatrix} = \begin{bmatrix} c+e & -(d+f) \\ d+f & c+e \end{bmatrix} \checkmark$$

$$k \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} kc & -(kd) \\ kd & kc \end{bmatrix} \checkmark$$

Let $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $B = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$

Then $A+B = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix}$

$$AB = \begin{bmatrix} \overset{Re(z)}{ac-bd} & -ad-bc \\ \underset{Im(z)}{ad+bc} & ac-bd \end{bmatrix}.$$

$$(a+bi)(c+di) = ac + adi + bci + bdi^2$$
$$= (ac-bd) + (ad+bc)i$$

$$\boxed{(a,b) \otimes (c,d) = (ac-bd,\ ad+bc)}$$

Find a basis for the span of the following vectors.

(a) $S = \{(1,1,1), (2,1,0), (0,1,1), (1,2,2)\}$
(b) $S = \{(3,1,-1,0), (0,-1,2,-1), (4,3,8,3)\}$
(c) $S = \{(1,1,2), (1,2,1), (-1,-7,4)\}$

$$\begin{bmatrix} 3 & 0 & 4 \\ 1 & -1 & 3 \\ -1 & 2 & 8 \\ 0 & -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

our vectors span the col(A).

OR

$$\begin{bmatrix} 3 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 4 & 3 & 8 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{bmatrix}$$

our vectors span row(A).