"Linear Algebra" - Systems of Linear Equations

$$\begin{cases} 3x - 7y = 6 \\ 2x + y = 7 \end{cases}$$

An ordered pair (x, y) is a solution to this system if it simultaneously satisfies both equations.

Solution Techniques: (1) Graph (intersection)
(2) Substitution
(3) Elimination

The theory of Linear Algebra is based on the process of solving systems via elimination.

$$\begin{cases} 3x - 7y = 6 \\ 2x + y = 7 \end{cases}$$

We use "elementary operations":

1. add one equation to another.
2. multiply an equation by a nonzero constant
3. swap equations

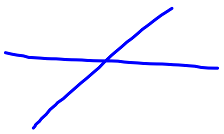
$$\begin{cases} 3x - 7y = 6 \\ 2x + y = 7 \end{cases} \xrightarrow[\text{(prop 2)}]{eq2 \cdot 7} \begin{cases} 3x - 7y = 6 \\ 14x + 7y = 49 \end{cases} \xrightarrow{eq1 + eq2}$$

$$\begin{cases} 17x + 0y = 55 \\ 14x + 7y = 49 \end{cases} \Longrightarrow x = \frac{55}{17}.$$

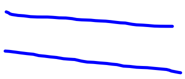
and $y = 7 - 2\left(\frac{55}{17}\right) = \frac{9}{17}$

$\therefore \left(\frac{55}{17}, \frac{9}{17}\right)$ is a solution.

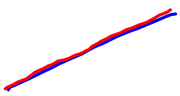
Possibilities (for linear systems):



$\leftarrow$ two lines cross once...
(one soln)



$\leftarrow$ do not cross
(no sols)



$\leftarrow$ coincident lines
(infinitely many sols)

$$\begin{cases} 2x+6y=4 \\ 3x+9y=7 \end{cases} \xrightarrow{\text{eq1}\cdot\frac{1}{2}} \begin{cases} x+3y=2 \\ 3x+9y=7 \end{cases} \xrightarrow{\text{eq2}\cdot\frac{1}{3}}$$

$$\begin{cases} x+3y=2 \\ x+3y=7/3 \end{cases} \; +(-\text{eq1}) \longrightarrow \begin{cases} x+3y=2 \\ 0x+0y=1/3 \end{cases}$$

algebraically: (no solution)

$$\boxed{0\cdot\text{variables} = \text{non-zero }\#}$$

geometrically: parallel lines

(we say the system is "inconsistent")

$$\begin{cases} 2x+6y=4 \\ 3x+9y=6 \end{cases} \xrightarrow{eq_2 - \frac{3}{2}eq_1} \begin{cases} 2x+6y=4 \\ 0x+0y=0 \end{cases}$$

$$\Rightarrow \begin{cases} y = \dfrac{4-2x}{6} \\ y = \dfrac{6-3x}{9} \end{cases}$$

$$\boxed{0 \cdot \text{variables} = 0}$$

algebraically: Any (all) ordered pairs would work (infinitely many solutions). (as long as they still satisfy the equation $2x+6y=4$).

geometrically: same (coincident) lines.

In cases where we have infinitely many sols, we often "parameterize" in terms of some other independent variable.

Since $2x+6y=4$, let $y=t$.

then $x = \dfrac{4-6t}{2}$

i.e. $(x, y) = (2-3t, t)$

leading variable ... "pivot"

$$\begin{cases} x + 2y + z = 4 \\ 3x + 8y + 7z = 20 & -3\,eq1 \\ 2x + 7y + 9z = 23 & -2\,eq1 \end{cases}$$

"Gaussian Elimination"

goal is to obtain triangular form:

$$\begin{array}{l} \_\_ + \_\_ + \_\_ = \_\_ \\ \quad\;\; \_\_ + \_\_ = \_\_ \\ \qquad\quad\;\; \_\_ = \_\_ \end{array}$$

or "echelon form"

$$= \begin{cases} x + 2y + z = 4 \\ \quad\; 2y + 4z = 8 & \cdot \frac{1}{2} \\ \quad\; 3y + 7z = 15 \end{cases}$$

"pivot"

$$= \begin{cases} x + 2y + z = 4 \\ \quad\; y + 2z = 4 \\ \quad\; 3y + 7z = 15 & -3\,eq2 \end{cases}$$

$$= \begin{cases} x + 2y + z = 4 \\ \quad\; y + 2z = 4 \\ \qquad\quad z = 3 \end{cases}$$

"pivot"

now we "back-substitute"

$z = 3$;

$y = 4 - 2z = 4 - 2(3) = -2$;

$x = 4 - 2(-2) - (3) = 5$

One unique soln: $(x, y, z) = (5, -2, 3)$.

$$\begin{cases} 3x - 8y + 10z = 22 \\ x - 3y + 2z = 5 \\ 2x - 9y - 8z = -11 \end{cases}$$

goal:
$$\begin{matrix} - & - & - & = & - \\ & - & - & = & - \\ & & - & = & - \end{matrix}$$

"upper triangular form"

$$= \begin{cases} x - 3y + 2z = 5 \\ 3x - 8y + 10z = 22 & -3\,eq\,1 \\ 2x - 9y - 8z = -11 & -2\,eq\,1 \end{cases}$$

$$= \begin{cases} x - 3y + 2z = 5 \\ y + 4z = 7 \\ -3y - 12z = -21 & +3\,eq\,2 \end{cases}$$

$$= \begin{cases} x - 3y + 2z = 5 \\ y + 4z = 7 \\ 0 = 0 \end{cases} \longrightarrow \text{infinitely many sols.}$$

Let $z = t$. Then $y = 7 - 4t$.

$$x = 5 + 3y - 2z$$
$$= 5 + 3(7-4t) - 2t$$
$$= 26 - 14t$$

$$\boxed{\text{So } (x, y, z) = (26 - 14t,\ 7 - 4t,\ t).}$$

Possible intersections of planes:

Image from http://geomalgorithms.com/a05-_intersect-1.html