# Linear Equations (some theory)

Focus on 1st order...

$$y'(t) + p(t)\,y(t) = f(t).$$

— dep. var and its derivative (plural if higher order)

— "coefficients" – functions of the independent variable $t$.

If $f(t)=0$, we have $y'(t)+p(t)y(t)=0$.
We call this a "homogeneous" equation.
Otherwise, if $f(t)\neq 0$, we have
$y'(t)+p(t)y(t)=f(t)$, called a "nonhomogeneous" equation.

Principles of linearity for homogeneous equations:

Suppose $y_1(t)$ and $y_2(t)$ both solve the homogeneous equation

$y' + p(t)y = 0$.

- Then any constant multiple of $y_1$ or $y_2$ would also solve the equation.
- The sum $y_1 + y_2$ solves the equation.

PROOFS - Let $y_1$ & $y_2$ be solutions of $y' + p(t)y = 0$. This implies

$\boxed{y_1' = -p(t)y_1}$ and $y_2' = -p(t)y_2$.

• Now, consider the function $ky_1$. Its derivative is $(ky_1)' = k(y_1)'$. Does this satisfy $y' + p(t)y = 0$?

$$k(y_1)' + p(t)\cdot ky_1 = k(\boxed{y_1'} + p(t)y_1)$$

this is where we used the hypothesis.

$$= k(\boxed{-p(t)y_1} + p(t)y_1)$$

$$= 0.$$

Proof of 2nd part:

Given $y_1$ & $y_2$ are solutions of $y' + p(t)y = 0$, we have

$$(y_1 + y_2)' = y_1' + y_2'$$
$$= -p(t)y_1 - p(t)y_2$$
$$= -p(t)(y_1 + y_2)$$

Then, substituting into the equation, we have:

$$(y_1 + y_2)' + p(t)(y_1 + y_2)$$
$$= -p(t)(y_1 + y_2) + p(t)(y_1 + y_2)$$
$$= 0.$$

$\therefore$ $y_1 + y_2$ Solves the homogeneous linear equation.

So... ① constant multiples of solutions to linear homogeneous equations are also solutions.

② sums of solutions to lin. homog. eqns. are also solutions.

(Linearity Principles for homogeneous equations)

(Principles of Superposition)

Ultimately, if $y_1, y_2, \ldots, y_n$ are all solutions to a lin. homog. eqn, then so is any solution of the form

$$y = a_1 y_1 + a_2 y_2 + \cdots + a_n y_n.$$

($a_i$ is constant).

## Linearity Principles for nonhomogeneous eqns.

Given a nonhomogeneous equation (linear)

$$y' + p(t)y = f(t) \quad (1)$$

we call

$$y' + p(t)y = 0 \quad (2)$$

its "associated homogeneous equation."

If $y_p$ solves (1) and $y_h$ solves (2),

then $y_p + y_h$ solves (1) as well.

p for "particular" h for "homogeneous"

(or sometimes $y_c$ for "complimentary")

PROOF:

Assume $y_p$ solves (1). This means

$$y_p' + p(t)y_p = f(t).$$

Also assume $y_h$ solves (2). This means

$$y_h' + p(t)y_h = 0.$$

Now, $y_h + y_p$ supposedly solves (1). Let's check:

$$(y_h + y_p)' + p(t)(y_h + y_p)$$

$$= y_h' + y_p' + p(t)y_h + p(t)y_p$$

$$= \underbrace{y_h' + p(t)y_h}_{=0 \text{ (b/c } y_h \text{ solves (2))}} + \underbrace{y_p' + p(t)y_p}_{= f(t) \text{ (b/c } y_p \text{ solves (1))}}$$

$$= f(t).$$

$\therefore$ $y_h + y_p$ solves the nonhomogeneous equation!

Finally, every solution to the linear nonhomogeneous equation is of the form $y = y_h + y_p$.

PROOF:

Suppose $y_p$ and $y_g$ are both solutions to the nonhomogeneous equation $y' + p(t)y = f(t)$, and $y_h$ is a solution to the associated homogeneous equation $y' + p(t)y = 0$.

Note that $\underline{y_g} = y_p + (y_g - y_p)$

this represents any solution to the nonhomogeneous eqn.

Note that

$$(y_g - y_p)' + p(t)(y_g - y_p)$$
$$= y_g' - y_p' + p(t)y_g - p(t)y_p$$
$$= \underbrace{y_g' + p(t)y_g}_{} - \underbrace{(y_p' + p(t)y_p)}_{}$$
$$= f(t) - f(t)$$
$$= 0$$

$\therefore$ $y_g - y_p$ is simply a solution to the homogeneous equation!

RECAP:

- Any particular solution $y_p$ can be written as $y_p = y_g + (y_p - y_g)$.
- But $y_p - y_g$ is just another $y_h$. (i.e., a homogeneous solution).
- So... we conclude that any particular solution $y_p$ is a sum of another particular solution $y_g$ and a homogeneous solution $y_h$... $y_p = y_g + y_h$.

$y'(t) = -0.5y + 10$ (nonhomogeneous)

$y(0) = 50$

$y'(t) = -0.5y$ (associated homogeneous eqn.)

$$\boxed{y_h(t) = Ce^{-0.5t}}$$

"transient solution" (homogeneous solution)

Now for $y_p$:

$y' + 0.5y = 10$ $\longrightarrow$ note that if $y = 20$, $y' = 0$

$0 + 0.5(20) = 10$

$$\boxed{\text{So } y_p = 20.}$$

"steady state solution"

$y = e^{at}$

$ae^{at} + 0.5e^{at} = 10$

$(a + 0.5)e^{at} = 10?$

$\Rightarrow a = 0$

$(0.5)e^0 \neq 10$

So, $y = y_h + y_p = Ce^{-0.5t} + 20.$

Now use $y(0) = 50$:

$$50 = Ce^{-0.5(0)} + 20$$

$$30 = C$$

$$\therefore \boxed{y = 30e^{-0.5t} + 20}$$

To solve a linear nonhomogeneous eqn:

(1) Use an integrating factor (yesterday), or

(2) use superposition...

- Find $y_h$.
- Find $y_p$.
- Create $y = y_h + y_p$
- Use initial conditions.

Look at notes for examples.

Ex:

$$\frac{dy}{dt} = -2y + e^t$$

"Guess" $y = e^t$ .... doh... no luck...

Better Guess: $y = Ae^t$. Try it in eqn.

$$(Ae^t)' = -2(Ae^t) + e^t$$

$$Ae^t = -2Ae^t + e^t$$

$$3Ae^t = e^t$$

$$\boxed{A = \frac{1}{3}}$$

$$\boxed{\text{So } y_p = \frac{1}{3}e^t.}$$

Now consider $y' = -2y$ (homogeneous)

$$y_h = Ce^{-2t}.$$

$$\therefore y(t) = y_h + y_p$$

$$= \boxed{Ce^{-2t} + \frac{1}{3}e^t.}$$

Ex:

$$\frac{dy}{dt} = -2y + \cos 3t$$

Let $y_p = A\cos 3t + B\sin 3t$. Then

$\frac{dy}{dt} + 2y = \cos 3t$ becomes

$$(-3A\sin 3t + 3B\cos 3t) + 2(A\cos 3t + B\sin 3t) = \cos 3t$$

$3B + 2A$ $\cos 3t$. one $\cos 3t$

$-3A + 2B$ $\sin 3t$. zero $\sin 3t$

$$\begin{cases} 2A + 3B = 1 \\ -3A + 2B = 0 \end{cases}$$

Solve for $(A, B) = \left(\frac{2}{13}, \frac{3}{13}\right)$, so

$$\boxed{y_p = \frac{2}{13}\cos 3t + \frac{3}{13}\sin 3t.}$$

$y_h$: $\frac{dy}{dt} = -2y \Longrightarrow \boxed{y_h = Ce^{-2t}.}$

$$y = y_h + y_p = \underbrace{Ce^{-2t}}_{\substack{\text{transient} \\ y_h}} + \underbrace{\frac{2}{13}\cos 3t + \frac{3}{13}\sin 3t}_{\substack{\text{steady state} \\ y_p}}.$$