Complex roots of the characteristic equation

Solve 3y'' - 7y' +2y = 0 with y(0) = 1, y'(0) = 2.

→ Let $y = e^{rt}$, then $y' = re^{rt}$

$y'' = r^2 e^{rt}$

$3r^2e^{rt} - 7re^{rt} + 2e^{rt} = 0$

$e^{rt}(3r^2 - 7r + 2) = 0$ → "characteristic equation"

$(r-2)(3r-1) = 0$

$r = 2$ or $r = \frac{1}{3}$. → "DISTINCT REAL ROOTS"

$\therefore \{y_1 = e^{2t},\ y_2 = e^{\frac{1}{3}t}\}$ ← fundamental set of solutions (if lin. ind.)

and $\boxed{y = c_1 e^{2t} + c_2 e^{\frac{1}{3}t}.}$ ← "general solution"

Use $y(0) = 1$ and $y'(0) = 2$...

$1 = c_1 e^0 + c_2 e^0$ $y' = 2c_1 e^{2t} + \frac{1}{3} c_2 e^{\frac{1}{3}t}$

$1 = c_1 + c_2$ $2 = 2c_1 + \frac{1}{3} c_2$

$c_1 + c_2 = 1$

$6c_1 + c_2 = 6$

$5c_1 = 5 \Rightarrow c_1 = 1,\ c_2 = 0$

$\boxed{\text{So } y = e^{2t}}$

Solve 9y'' - 12y' + 4y = 0.

$9 \times 4 = 36$; what mults to give 36 and adds to give 12?

6 & 6

$$9r^2 - 12r + 4 = 0$$

$$\underbrace{9r^2 - 6r}\underbrace{-6r + 4} = 0$$

$$\underbrace{3r(3r-2)}-\underbrace{2(3r-2)} = 0$$

$$(3r-2)(3r-2) = 0$$

"REPEATED REAL ROOTS"

$$r_1 = r_2 = r = \frac{2}{3}.$$

$$\left\{ y_1 = e^{\frac{2}{3}t},\ y_2 = te^{\frac{2}{3}t} \right\}$$ fundamental solution set.

$$\boxed{y = c_1 e^{\frac{2}{3}t} + c_2 t e^{\frac{2}{3}t}.}$$

Consider a sring/mass/dashpot system in which m = 1, b = 2 (damping constant), k = 5 (spring constant). Solve the corresponding equation my'' + by' + ky = y'' + 2y' +5y = 0.

$$\underbrace{r^2+2r}+5=0$$

$$(r+1)^2+4=0 \rightarrow (r+1)^2=-4$$

$$r+1=\sqrt{-4}$$

$$\boxed{r=-1\pm 2i} \qquad r=-1\pm 2i$$

"Complex conjugate roots"

$$\left\{y_1=e^{(-1+2i)t},\ y_2=e^{(-1-2i)t}\right\}$$ fundamental solution set

However, $y=c_1e^{(-1+2i)t}+c_2e^{(-1-2i)t}$

is "complex-valued;" i.e., its value in general could be a complex number, and how might we interpret that in physical terms??

Note that $e^{(a+bi)t} = e^{at+ibt} = e^{at}e^{ibt}$, and recall a Taylor Series for e^t:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

How about a Taylor Series for e^{ibt}?

We have $e^{(a+bi)t} = e^{at+bti} = \underbrace{e^{at}}_{\mathbb{R}}\underbrace{e^{bti}}_{\mathbb{C}}$

Let $\theta = bt$.

$$e^{(i\theta)} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$$\Longrightarrow \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

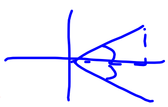
Euler's Formula

FUN FACT: $\left(e^{i\pi} = \cos\pi + i\sin\pi = -1 \right)$

$$\boxed{e^{i\pi} + 1 = 0}$$

So $y_1 = e^{(a+bi)t} = e^{at}e^{ibt} = e^{at}(\cos(bt) + i\sin(bt))$

And $y_2 = e^{(a-bi)t} = e^{at}e^{-ibt} = e^{at}(\cos(-bt) + i\sin(-bt))$



$$= e^{at}(\cos(bt) - i\sin(bt))$$

$\therefore$

$$y = \underbrace{C_1 e^{at}(\cos bt + i\sin bt)}_{y_1} + \underbrace{C_2 e^{at}(\cos bt - i\sin bt)}_{y_2}.$$

☹ ... still complex-valued ...

If f(t) = u(t) + v(t)i is a solution to
ay'' + by' + cy = 0:

$$f'(t) = u'(t) + i\,v'(t).$$

$$f''(t) = u''(t) + i\,v''(t)$$

$$a(u'' + iv'') + b(u' + iv') + c(u + iv) = 0$$

$$\underbrace{(au'' + bu' + cu)}_{\text{Real}} + i\underbrace{(av'' + bv' + v)}_{\text{Imaginary}} = 0$$

If $a + bi = 0$, $a = 0 = b$.

$\Rightarrow$ $au'' + bu' + cu = 0$ $\Rightarrow$ $u(t)$ and $v(t)$ are each solutions!

and $av'' + bv' + cv = 0$

Summary... If $f(t) = u(t) + iv(t)$

is a solution to

$$ay'' + by' + cy = 0,$$

then $u(t)$ (the real part of f)

and $v(t)$ (the imagnary part of f)

are also both solutions, and they're

both real-valued!!

$$y'' + 2y' + 5y = 0$$

$$\vdots$$

$$r = -1 \pm 2i$$

pick <u>one</u> of the complex roots:

$$r = -1 + 2i$$

$$e^{(-1+2i)t} = e^{-t} e^{2it} = e^{-t}(\cos 2t + i \sin 2t)$$

$$= \underbrace{e^{-t}\cos 2t}_{u(t)} + i\,\underbrace{e^{-t}\sin 2t}_{v(t)}$$

Then $y_1 = e^{-t}\cos 2t$ $y_2 = e^{-t}\sin 2t$.

And $y = c_1 y_1 + c_2 y_2$

$$\boxed{y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t}$$

Real-valued!

Try:

$$y'' + 4y' + 9y = 0$$

$$r^2 + 4r + 9 = 0$$

$$(r+2)^2 + 5 = 0$$

$$r = -2 \pm i\sqrt{5}$$

$$\cancel{y_1 = e^{(-2+i\sqrt{5})t} = e^{-2t}(\cos\sqrt{5}t + i\sin\sqrt{5}t)}$$

$$\cancel{y_2 = e^{(-2-i\sqrt{5})t} = e^{-2t}(\cos\sqrt{5}t - i\sin\sqrt{5}t)}$$

$$y_1 = e^{-2t}\cos(\sqrt{5}t) \qquad y_2 = e^{-2t}\sin(\sqrt{5}t)$$

$$y = e^{-2t}(c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t)$$

What if the characteristic equation leads to

(1) $r_1 \neq r_2$ (real) $\quad e^{r_1 t} \quad e^{r_2 t}$

(2) $r_1 = r_2$ (real) $\quad e^{r_1 t} \quad te^{r_1 t}$

(3) $r = a \pm bi$ (complex)

$$e^{at}(\cos bt) \quad e^{at}(\sin bt)$$

Ex: $(r-2)^2(r+1)^3(r) = 0$

alg. multiplicity of 2...

$$y_1 = e^{2t}, \quad y_2 = te^{2t}$$

$$y_3 = e^{-t}, \quad y_4 = te^{-t}, \quad y_5 = t^2e^{-t}$$

$$y_6 = e^{0t} = 1.$$

$$\Rightarrow y(t) = c_1e^{2t} + c_2te^{2t} + c_3e^{-t} + c_4te^{-t} + c_5t^2e^{-t} + c_6.$$

$r = 0, 0, 0, 0, -4, 3 \pm 2i, 3 \pm 2i$

$0: \quad e^{0t}(c_1 + c_2 t + c_3 t^2 + c_4 t^3)$

$-4: \quad c_5 e^{-4t}$

$3+2i: \quad c_6 e^{3t}\cos 2t + c_7 t e^{3t}\cos 2t + c_8 e^{3t}\sin 2t + c_9 t e^{3t}\sin 2t$

OR

$(c_6 + c_7 t)e^{3t}\cos 2t + (c_8 + c_9 t)e^{3t}\sin 2t$