Section 4.2 #s 1–14

W is the set of all vectors in $\mathbb{R}^3$ such that $x_1 = 5x_2$. → (1st comp. is 5 times the 2nd comp.)

Is W a subspace of $\mathbb{R}^3$?

Let $\vec{u} = (a, b, c)$ and $\vec{v} = (x, y, z)$ be vectors in W, and let $k \in \mathbb{R}$ be a scalar.

What do we mean when we say $\vec{u}$ and $\vec{v}$ are in W?

$$\vec{u} = (a, b, c) = (5b, b, c)$$

$$\vec{v} = (x, y, z) = (5y, y, z)$$

$$\vec{u} + \vec{v} = (5b+5y,\ b+y,\ c+z)$$

- Is the 1st comp. still 5 times the 2nd?

$= (5(b+y),\ b+y,\ c+z)$ YES ✓

$$k\vec{u} = (k \cdot 5b,\ k \cdot b,\ k \cdot c)$$

- Is the 1st comp. still 5 times the 2nd?

$= (5(kb),\ kb,\ kc)$ YES ✓.

∴ The set W is a subspace of $\mathbb{R}^3$.

Let W be the set of all vectors in $\mathbb{R}^3$ such that $x_1 + x_2 + x_3 = 2$.

(i.e., the sum of the components is 2)

Let $\vec{u}, \vec{v} \in W$, and $k \in \mathbb{R}$.

$\vec{u} = (u_1, u_2, u_3)$

$\vec{v} = (v_1, v_2, v_3)$

Then $u_1 + u_2 + u_3 = 2$.

$v_1 + v_2 + v_3 = 2$.

$\vec{u} + \vec{v} = (u_1 + v_1,\ u_2 + v_2,\ u_3 + v_3)$

- Do the components add to 2?

$(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3)$

$= (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3)$

$= 2 + 2$

$= 4$

No :(

(W is not closed under vector add'n)

i.e., W is not a subspace.

Scalar mult?

Is $k\vec{u}$ still in W?

$k\vec{u} = (ku_1, ku_2, ku_3)$

Do these components add to 2?

$ku_1 + ku_2 + ku_3 = k(u_1 + u_2 + u_3)$

$= k \cdot 2$

(not equal to 2 for all k, so W is not closed under scalar mult, i.e., W is not a subspace.)

W is the set of all vectors in $\mathbb{R}^4$ such that $x_1 = 3x_3$ and $x_2 = 4x_4$.

Is W a subspace of $\mathbb{R}^4$?

Let $\vec{u} = (a, b, c, d)$; $\vec{v} = (p, q, r, s)$ be vectors in W and k a scalar.

$\vec{u} \in W \Rightarrow (3c, 4d, c, d) = \vec{u}$

$\vec{v} \in W \Rightarrow \vec{v} = (3r, 4s, r, s)$

Is $\vec{u} + \vec{v} \in W$?

$\vec{u} + \vec{v} = (3(c+r), 4(d+s), c+r, d+s) \in W$

(closed under addition)

Is $k\vec{u} \in W$?

$k\vec{u} = (k(3c), k(4d), kc, kd)$

$= (3(kc), 4(kd), kc, kd) \in W$

(closed under scalar mult.)

$\therefore$ W is a subspace of $\mathbb{R}^4$.

W is the set of all vectors in $\mathbb{R}^4$ such that $x_1+2x_2+3x_3+4x_4=0$.

Is W a subspace of $\mathbb{R}^4$?

Let $\vec{u}=(u_1,u_2,u_3,u_4) \in W$

$\vec{v}=(v_1,v_2,v_3,v_4) \in W$

$k \in \mathbb{R}$ (scalar)

So $u_1+2u_2+3u_3+4u_4=0$

$v_1+2v_2+3v_3+4v_4=0$

$\vec{u}+\vec{v}=(u_1+v_1,\ u_2+v_2,\ u_3+v_3,\ u_4+v_4)$

Does this vector satisfy the eqn?

$(u_1+v_1)+2(u_2+v_2)+3(u_3+v_3)+4(u_4+v_4)=$

$(u_1+2u_2+3u_3+4u_4)+(v_1+2v_2+3v_3+4v_4)=$

$(0) \quad + \quad (0) \quad = 0.$

(W is closed under addn)

Does $k\vec{u}$ satisfy the eqn?

$k\vec{u}=(ku_1,ku_2,ku_3,ku_4)$

$ku_1+2(ku_2)+3(ku_3)+4(ku_4)$

$=k(u_1+2u_2+3u_3+4u_4)$

$=k(0)$

$=0.$ (W is closed under scalar mult.)

Thm:

If A is a constant $m \times n$ matrix then the solution set of the homogeneous linear system $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

Proof:

A $m \times n$ matrix A yields the homogeneous system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

Note that any solution will be of the form $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Now take two solutions from the set of all possible solutions, say $\vec{x}_0$ & $\vec{y}_0$ and k a scalar.

If $\vec{x}_0$ is a solution then $A\vec{x}_0 = \vec{0}$.

If $\vec{y}_0$ is a solution then $A\vec{y}_0 = \vec{0}$.

- Is $\vec{x}_0 + \vec{y}_0$ a solution?

  Note that $A(\vec{x}_0 + \vec{y}_0) = A\vec{x}_0 + A\vec{y}_0$

  matrix mult. is distrib. $= \vec{0} + \vec{0} = \vec{0}$

  Since $A(\vec{x}_0 + \vec{y}_0) = \vec{0}$, $\vec{x}_0 + \vec{y}_0$ is also in the set of solutions to $A\vec{x} = \vec{0}$, (i.e., the solution is closed under addition)

- Is $k\vec{x}_0$ a solution?

  Note that $A(k\vec{x}_0) = kA\vec{x}_0$

  $= k(A\vec{x}_0) = k(\vec{0}) = \vec{0}$.

So $k\vec{x}_0$ is also a solution!

$\therefore$ The set of all solutions to $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

(we call this the "solution space" of the system.)

Let W be the set of all 2×2 diagonal matrices. Is this a subspace of $M_{2\times 2}$ $\leftarrow$ (set of all 2×2 matrices)

Let $\vec{u} = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix} \in W$

and $\vec{v} = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix} \in W$

and k a scalar,

- Is $\vec{u}+\vec{v} \in W$?
- Is $k\vec{u} \in W$?