Common questions:

1) Determine if the vectors u, v, ..., and w are linearly independent.

2) Express the vector t as a linear combination of u, v, ..., and w

3) Find scalars a, b, ..., c such that a**u** + b**v** + ... + c**w** = 0

(Ex 5, pg. 245) Determine whether the vectors $\vec{v}_1 = (1,2,2,1)$

$\vec{v}_2 = (2,3,4,1)$

$\vec{v}_3 = (3,8,7,5)$

are lin. dep. or linearly ind.

ANS: Does $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3$ have any non-trivial solns?

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 8 & 0 \\ 2 & 4 & 7 & 0 \\ 1 & 1 & 5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

3 vectors in $\mathbb{R}^4$.

We know we have the trivial soln, and we have no free variables.

$\therefore$ the vectors are linearly independent.

$\vec{u} = (2,0,1) \quad \vec{v} = (-3,1,-1) \quad \vec{w} = (0,-2,-1).$

Lin. ind. or dep.?

Is there a non-trivial soln to $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$?

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable and homogeneous sys, so

Linearly dependent b/c there are non-trivial sols.

3) Find scalars a, b, ..., c such that a**u** + b**v** + ... + c**w** = **0**

c is free... $b = 2c$, $a = 3c$.

Let $c = \pi$ then

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$$

$$\Longrightarrow \boxed{3\pi\vec{u} + 2\pi\vec{v} + \pi\vec{w} = \vec{0}.}$$

2) Express the vector u as a linear combination of v and w

$\vec{u} = (2,0,1) \qquad \vec{v} = (-3,1,-1) \qquad \vec{w} = (0,-2,-1)$.

$$a\vec{v} + b\vec{w} = \vec{u}$$

$$\iff \left(a\vec{v} + b\vec{w} - 1\vec{u} = \vec{0} \right)$$

("same" approach as lin. ind.)

$$a\begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$\vdots$

$$\begin{bmatrix} -3 & 0 \\ 1 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -3 & 0 & 2 \\ 1 & -2 & 0 \\ -1 & -1 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & -2/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\vec{u} = \frac{-2}{3}\vec{v} + \left(-\frac{1}{3}\right)\vec{w}}$$

A vector $\vec{v}$ in a vector space can be uniquely expressed as a linear combination of linearly independent vectors.

Assume $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are lin. ind. vectors in the space V.

Suppose $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$

and $\vec{w} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_k\vec{v}_k$

Subtract:

$$\vec{0} = (a_1 - b_1)\vec{v}_1 + (a_2 - b_2)\vec{v}_2 + \cdots + (a_k - b_k)\vec{v}_k$$

Since the $\vec{v}_i$ are lin. ind., this system has only the trivial soln. So

$$a_1 - b_1 = 0 \implies a_1 = b_1$$

$$a_2 - b_2 = 0 \qquad a_2 = b_2$$

$$\vdots \qquad \vdots$$

$$a_k - b_k = 0 \qquad a_k = b_k.$$

Prove $\vec{0}$ is in <u>every</u> subspace.

WRONG: $k\vec{0} = \vec{0}$, $\therefore$ $\vec{0} \in$ every subspace.

INWRONG: Let $\vec{v}$ be a vector in the subspace. Then $k\vec{v}$ is in the subspace (b/c subspaces are closed under scalar mult.) $\therefore$ $0\vec{v} = \vec{0}$ is in the subspace.

Nature of Sols to Linear Systems

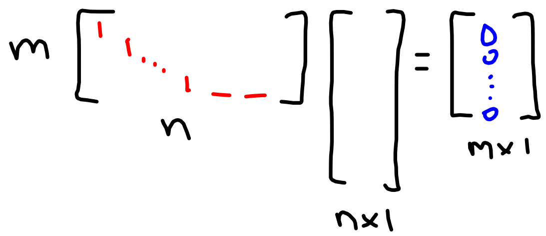
$A\vec{x} = \vec{0}$ (homogeneous)

We always have at least the trivial soln!

$n \times n$ (# rows = # cols) $\Leftrightarrow$ (# vars = # eqns)

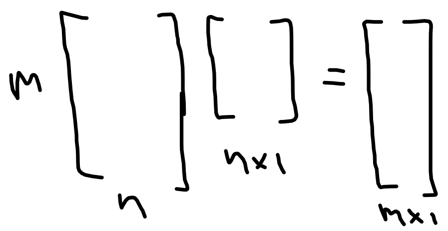
- A row equiv. to I (or A^{-1} exists, $|A| \neq 0$) $\Rightarrow$ only trivial soln.
- A not row equiv. to I $\Rightarrow$ free vars, so infinitely many sol'ns

vars > # eqns $\quad m < n$



- We have the trivial soln.
- We will always have free vars, so we will always have infinitely many solutions.

vars < # eqns $\quad m > n$



- we have the trivial soln
- we could have either one unique soln (trivial soln) or infinitely many sol's.

$$\begin{cases} x + y = 0 \\ 2x - 3y = 0 \\ 4x + 2y = 0 \end{cases} \qquad \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & -5 \\ 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

One soln.

vs

$$\begin{cases} x + y = 0 \\ 2x + 2y = 0 \\ 3x + 3y = 0 \end{cases} \qquad \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\leftarrow$ free var, inf. many sols.

Nature of Sols to Linear Systems

$A\vec{x} = \vec{b}$ $\quad \vec{b} \neq \vec{0}$ (non-homogeneous).

$$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$$

$n \times n$ (# rows = # cols) $\Leftrightarrow$ (# vars = # eqns) We never have the trivial soln!

- A row equiv. to I (or A^{-1} exists, $|A| \neq 0$) $\Rightarrow$ only one (unique) soln.
- A not row equiv. to I $\Rightarrow$ free vars, so infinitely many solns, or no sols.

 # vars > # eqns $\quad m < n$

$$m\underbrace{\begin{bmatrix} 1 & & & \\ & 1 \ddots & & \\ & & 1 & - - \end{bmatrix}}_{n} \underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{m \times 1}$$

$$\begin{cases} x+y+z=3 \\ 2x+2y+2z=6 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x+y+z=3 \\ 2x+2y+2z=5 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

- We will always have free vars, so we will have inf. many sols. if we have a consistent system. (or no sols.)

 # vars < # eqns $\quad m > n$

$$m\underbrace{\begin{bmatrix} & \\ & \\ & \end{bmatrix}}_{n} \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{m\times 1}$$

- We could have either one unique soln or infinitely many sol's.

$$\begin{cases} x+y=1 \\ 2x+2y=7 \\ 3x+3y=4 \end{cases} \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 7 \\ 3 & 3 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{array}\right]$$

no sols.

vs

$$\begin{cases} x+y=1 \\ 2x+2y=2 \\ 3x+3y=3 \end{cases} \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

free var, inf. many sols.

vs

$$\begin{cases} x+y=1 \\ 2x+3y=2 \\ 3x+4y=3 \end{cases} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 3 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

unique soln.