A vector space V is a set of objects on which addition and scalar multiplication are defined in a way that conditions (a) - (h) on page 225-226 are met.

Examples of vector spaces:

- $n \times n$ matrices
- functions continuous on $(-\infty, \infty)$
- functions with continuous derivatives
- polynomials

Consider functions continuous on $(-\infty, \infty)$.
This is a vector space (prove a-h).

Let W be the set of all polynomials, which are continuous on $(-\infty, \infty)$.
They "inherit" properties a-h from the set of functions continuous on $\mathbb{R}$.
W is a <u>subspace</u> because:

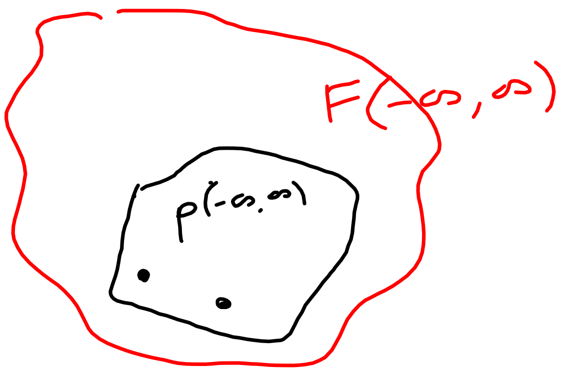
If $p(x)$ and $q(x)$ are polynomials and k is some scalar, then:

(1) $p(x) + q(x) = r(x)$ (another poly).

(i.e., polynomials are <u>closed</u> under addition).

(2) $k \cdot p(x) = u(x)$ (another poly.)

(i.e., polynomials are <u>closed</u> under scalar multiplication)



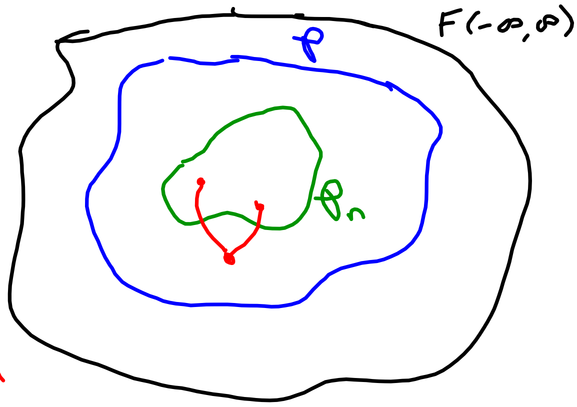
Are __nth__ degree polynomials a subspace?

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$

$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_2 x^2 + b_1 x + b_0$

$(a_n, b_n \neq 0)$

and k a scalar.



$p(x) + q(x)$ does <u>not</u> always yield an nth degree poly, so the set of all nth-degree polynomials is <u>not</u> closed under addition (or scalar multiplication) (i.e., it's <u>not</u> a subspace.

However, $\mathcal{P}_n$, the set of all polynomials of degree n or less, <u>is</u> a subspace.

Consider 3×3 symmetric matrices.

Symmetric if $A^T = A$.

example $\begin{bmatrix} 1 & -2 & 13 \\ -2 & 3 & -5 \\ 13 & -5 & 7 \end{bmatrix}$

Let $A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$, $B = \begin{bmatrix} x & w & v \\ w & y & u \\ v & u & z \end{bmatrix}$

Is $A+B$ symmetric?

$$A+B = \begin{bmatrix} a+x & d+w & e+v \\ d+w & b+y & u+f \\ e+v & u+f & c+z \end{bmatrix}$$ yes!

$$kA = \begin{bmatrix} ka & kd & ke \\ kd & kb & kf \\ ke & kf & kc \end{bmatrix}$$ yes!

$\therefore$ The set of all 3×3 symmetric matrices forms a subspace!

$\mathbb{R}^3 \longrightarrow$ Subspaces:

$\mathbb{R}^3$

Planes through origin

Lines through origin

$\{\vec{0}\}$

"proper subspaces"

"trivial"... b/c $\vec{0}$ is a subspace of any vector space, and the entire space is a subspace of itself.

Linear Independence of Vectors in $\mathbb{R}^2$ & $\mathbb{R}^3$

- The vectors $\vec{u}$ & $\vec{v}$ are linearly dependent if there exist scalars a, b not both zero, such that

$$a\vec{u} + b\vec{v} = \vec{0}. \quad \left(\text{i.e., } \vec{u} = k\vec{v}.\right)$$

OR the vectors $\vec{u}$ & $\vec{v}$ are linearly independent if the only solution to $a\vec{u} + b\vec{v} = \vec{0}$ is $a = b = 0$.

In $\mathbb{R}^3$:

If $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0} \implies a, b, c = 0$, ($\vec{u}, \vec{v}, \vec{w}$ are linearly independent.)

Note: In $\mathbb{R}^3$ (for example), the 3 vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent if $|A| = \begin{vmatrix} \vec{u} & \vec{v} & \vec{w} \end{vmatrix} \neq 0$.

<u>Why</u>: If $\vec{u}, \vec{v}, \vec{w}$ are lin. ind., then

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$$

has <u>only</u> the trivial solution $a=b=c=0$.

$$a\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + b\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + c\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\vdots$

$$\left[\begin{array}{ccc|c} u_1 & v_1 & w_1 & 0 \\ u_2 & v_2 & w_2 & 0 \\ u_3 & v_3 & w_3 & 0 \end{array}\right]$$

If $A \to I$ (A^{-1} exists, $|A| \neq 0$), we have <u>only</u> the trivial solution

Are $\vec{u} = (1,2,-3)$, $\vec{v} = (3,1,-2)$, $\vec{w} = (5,-5,6)$ linearly independent?

We need to see if $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$ has only the trivial solution.

$$a\begin{bmatrix}1\\2\\-3\end{bmatrix} + b\begin{bmatrix}3\\1\\-2\end{bmatrix} + c\begin{bmatrix}5\\-5\\6\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 3 & 5\\2 & 1 & -5\\-3 & -2 & 6\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1 & 3 & 5 & 0\\2 & 1 & -5 & 0\\-3 & -2 & 6 & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 3 & 5 & 0\\0 & 1 & 3 & 0\\0 & 0 & 0 & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 0 & -4 & 0\\0 & 1 & 3 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

- A is not row equiv. to I.
- A is not invertible
- $|A|$ is 0.
- $A\vec{x} = \vec{0}$ has non-trivial sols.

If the vectors are dependent, determine scalars a, b, c such that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$.

(i.e., find a "linear combination" of $\vec{u}, \vec{v}, \vec{w}$ that yields $\vec{0}$.)

$b = -3c$, $a = 4c$

So... any $(a,b,c) = (4c, -3c, c)$ works!

$(4, -3, 1)$

$4\vec{u} - 3\vec{v} + \vec{w} = \vec{0}$.

Linear dependence $\Longrightarrow$ One vector can be written as a linear combination of the others.

Note:

$$\vec{w} = 3\vec{v} - 4\vec{u}$$

$$\vec{u} = \frac{3}{4}\vec{v} - \frac{1}{4}\vec{w}$$

$$\vec{v} = \frac{4}{3}\vec{u} + \frac{1}{3}\vec{w}$$

If A is $m \times n$, then the solution set of the homogeneous linear system $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

$$m\left[\begin{array}{ccc} & & \\ & & \\ & & \end{array}\right]_{n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

The $\vec{x}$ (the solution vector) has n components, so $\vec{x} \in \mathbb{R}^n$.

Closure?

Assume $\vec{u}, \vec{v}$ are solutions to $A\vec{x} = \vec{0}$, and $k \in \mathbb{R}$ is a scalar.

Is $\vec{u} + \vec{v}$ a solution to $A\vec{x} = \vec{0}$?

$$\begin{aligned} A(\vec{u} + \vec{v}) &= A\vec{u} + A\vec{v} \\ &= \vec{0} + \vec{0} \\ &= \vec{0} \end{aligned}$$

b/c $\vec{u}, \vec{v}$ are sols.

Is $k\vec{u}$ a solution to $A\vec{x} = \vec{0}$?

$$\begin{aligned} A(k\vec{u}) &= k(A\vec{u}) \\ &= k\vec{0} \\ &= \vec{0} \end{aligned}$$