3.6 - **Determinants** of n x n matrices

What is a determinant? A determinant is a function that takes an n x n matrix as input and yields a single value as its output. (Specifically, if the elements of the matrix are all real numbers, the determinant will be a real number y such that det(A) = y.)

Interestingly enough, while determinants are studied and defined almost exclusively in terms of matrices, their development predated matrices by about 200 years!

The definition above doesn't really tell us how to calculate a determinant. Determinants arose out of some general observations from the study of systems of equations. For a nice discussion of how it all plays out eventually, see pages 93-98 of "Paul's Online Linear Algebra Notes" (link on webpage).

While ALL of linear algebra can be done without determinants, we make use of them to understand a few important ideas:

1) Test for invertibility of matrices

2) Calculating Eigenvales and Eigenvectors

3) Calculating the Wronskian

4) Calculating the Jacobian

So - perhaps this will help: The determinant of an n x n matrix A, denoted det(A) or |A|, is the sum of the signed elementary products of the n x n matrix A.

Nope. Doesn't help. What is a "signed elementary product?"

Consider the following 2 x 2 matrix: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

An "elementary product" of an n x n matrix A is a product of n factors - exactly one factor from each row and one factor from each column (implying that we cannot use two factors from any given row or any given column). In other words, if you're building an elementary product for the 2 x 2 matrix above, once you grab a factor from row 1, the other factor(s) cannot come from row 1.

Subject to these rules, list all the possible elementary products of the matrix above.

You should have found that the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ leads to the elementary products $a_{11}a_{22}$ and $a_{12}a_{21}$.

But what is this "signed" business?

Note that as I've written the products, the row indices are always in the order (1,2). However, the column indices vary ... the first product has column indices (1,2) and the second product has column indices (2,1). In the latter case, we call this a "permutation" of the column indices, and to generate this particular permutation of (1,2), exactly one pair of numbers has been "inverted." In the former case, no pairs have been inverted. Note that "0" is even and "1" is odd.

In general, if the total number of inversions of row and column indices in a particular permutation is even, then the elementary product is assigned a "+" sign, whereas if the total number of inversions is odd, the elementary product is assigned a "-" sign.

Since I wrote the products in such a way that the row indices are not inverted at all, the number of inversions of row and columns really just boils down to the number of inversions of just the columns - but still - remember that it's the SUM that counts.

So - according to the definition, the determinant of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is the sum of the signed elementary products $a_{11}a_{22}$ and $a_{12}a_{21}$.

The first product has zero inversions (even, so a sign of "+") and the second has one inversion (odd, so a sign of "-"). The determinant is therefore

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Now consider a 3 x 3 matrix $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$.

Determine all of the elementary products. It's helpful to be systematic and keep the rows in their original orders and just work with inversions of the columns:

$b_{11}\ b_{22}\ b_{33}$

$b_{11}\ b_{23}\ b_{32}$

$b_{1\_}\ b_{2\_}\ b_{3\_}$

$b_{1\_}\ b_{2\_}\ b_{3\_}$

etc. ...

How many possibilities are there? (This is an easy combinatorics problem.) Write out all of the elementary products for matrix B.

For the 3 x 3 matrix $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$, you should have found the following elementary products:

$b_{11}\ b_{22}\ b_{33}$	$b_{12}\ b_{21}\ b_{33}$	$b_{13}\ b_{21}\ b_{32}$
$b_{11}\ b_{23}\ b_{32}$	$b_{12}\ b_{23}\ b_{31}$	$b_{13}\ b_{22}\ b_{31}$

Now, determine the "sign" of each elementary product. In other words, count how many "inversions" of column indices occurred in each (since we kept the row indices in the order (1,2,3) for the entire list). The quickest way to do this is to count how many times a smaller index occurs to the right of a larger index.

permutation	inversions	# of inversions	sign
$b_{11}b_{22}b_{33}$			
$b_{12}b_{23}b_{31}$			
$b_{13}b_{21}b_{32}$			
$b_{12}b_{21}b_{33}$			
$b_{11}b_{23}b_{32}$			
$b_{13}b_{22}b_{31}$			

$+ b_{11} b_{22} b_{33} \quad - b_{12} b_{21} b_{33}$

$- b_{11} b_{23} b_{32} \quad + b_{12} b_{23} b_{31}$

$b_{1\_}\ b_{2\_}\ b_{3\_} \quad + b_{13} b_{21} b_{32}$

$b_{1\_}\ b_{2\_}\ b_{3\_} \quad - b_{13} b_{22} b_{31}$

etc. ...

Therefore, given the matrix $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

We calculate the determinant as

det(B) =

permutation	inversions	# of inversions	sign
$b_{11}b_{22}b_{33}$			
$b_{12}b_{23}b_{31}$			
$b_{13}b_{21}b_{32}$			
$b_{12}b_{21}b_{33}$			
$b_{11}b_{23}b_{32}$			
$b_{13}b_{22}b_{31}$			

$+\, b_{11}\, b_{22}\, b_{33}$ $\quad -\, b_{12} b_{21} b_{33}$

$-\, b_{11}\, b_{23}\, b_{32}$ $\quad +\, b_{12} b_{23} b_{31}$

$b_{1\_}\ b_{2\_}\ b_{3\_}$ $\quad +\, b_{13} b_{21} b_{32}$

$b_{1\_}\ b_{2\_}\ b_{3\_}$ $\quad -\, b_{13} b_{22} b_{31}$

etc. ...

Therefore, given the matrix $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{matrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{matrix}$

$- \quad - \quad -$

$+ \quad + \quad +$

We calculate the determinant as

det(B) =

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ -2 & -2 & 1 \end{bmatrix}$$

$$\det B = |B| = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ -2 & -2 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 3 & 0 \\ -2 & -2 \end{matrix}$$

$$= (1)(0)(1) + (2)(6)(-2) + (4)(3)(-2) - (2)(3)(1) - (1)(6)(-2) - (4)(0)(-2)$$

$$= -24 - 24 - 6 + 12$$

$$= -42\ ?$$

We have shortcuts for 2 x 2 and 3 x 3:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{matrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{matrix}$$

$$\det\left(\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}\right) = (4)(1) - (3)(-2)$$

$$= 10.$$

No such luck for n x n in general ... instead, we have "minors" and "cofactors."

MINORS and COFACTORS

The *ij*th minor of a matrix A (or the minor of a_{ij}) is the determinant M_{ij} of the (n-1) x (n-1) submatrix that remains when row i and column j are removed from A.

The *ij*th cofactor A_{ij} of A (or the cofactor of a_{ij}) is defined as

$A_{ij} = (-1)^{i+j}M_{ij}$

The determinant det(A) of the n x n matrix A is defined as:

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

In other words, the determinant of a matrix A is the sum of each entry in a given row (or column) times its corresponding cofactor.

$$C = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 6 \\ -2 & -2 & 1 \end{bmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$\left(\text{each } A_{ij} = (-1)^{i+j} M_{ij}\right)$$

Using a minor & cofactor expansion to calculate $\det(C)$:

Suppose we expand along row 1: sign:

$a_{11} = 1$. $M_{11} =$ det $= \begin{vmatrix} 2 & 6 \\ -2 & 1 \end{vmatrix} = 2-(-12) = 14$. $(-1)^{1+1} = 1$.

$a_{12} = 2$. $M_{12} = \begin{vmatrix} 3 & 6 \\ -2 & 1 \end{vmatrix} = 3-(-12) = 15$. $(-1)^{1+2} = -1$

$a_{13} = 4$. $M_{13} = \begin{vmatrix} 3 & 2 \\ -2 & -2 \end{vmatrix} = -6-(-4) = -2$. $(-1)^{1+3} = 1$.

So $\det(C) = +1(14) + (-1)(15)(2) + 4(-2)(+1) = -24$.

$$C = \begin{bmatrix} 1 & {}^{-}2 & 4 \\ 3 & {}^{+}2 & 6 \\ -2 & {}^{-}{-2} & 1 \end{bmatrix}$$

$$\begin{matrix} + & - & + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & \ddots \\ - & \vdots \\ \vdots \end{matrix}$$

Expansion along column 2.

$$\det(C) = -2\begin{vmatrix} 3 & 6 \\ -2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 4 \\ -2 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix}$$

$$= -2(3+12) + 2(1+8) + 2(-6)$$

$$= -30 + 18 - 12$$

$$= -24.$$

BE Smart.

$$\begin{vmatrix} 4 & -147.2 & 2 \\ 3 & 0 & 0 \\ 2 & 3 & 1 \end{vmatrix} = -3(-147.2-6) + 0\left(\substack{\text{who}\\\text{cares}}\right) - 0\left(\substack{\text{who}\\\text{cares}}\right)$$

$$\begin{bmatrix} 4 & 2 & -1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 1 & 4 & 6 \\ 1 & 3 & 3 & -1 \end{bmatrix}$$

Row 4:

$$(-1)\begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & 1 \\ 1 & 4 & 6 \end{vmatrix} + 3\begin{vmatrix} 4 & -1 & 2 \\ 2 & 2 & 1 \\ 3 & 4 & 6 \end{vmatrix} - 3\underbrace{\cancel{\begin{vmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 3 & 1 & 6 \end{vmatrix}}}_{=0} + (-1)\begin{vmatrix} 4 & 2 & -1 \\ 2 & 1 & 2 \\ 3 & 1 & 4 \end{vmatrix}$$

$$(-1)\left[2(12-4)-(-1)(6-1)+2(4-2)\right]$$

$$+(3)\left[4(12-4)-(-1)(12-3)+2(8-6)\right]$$

$$-(3)\left[0 \text{ b/c one row or col. is a multiple of another}\right]$$

$$+(-1)\left[4(4-2)-2(8-6)+(-1)(2-3)\right]$$

$$=(-1)\left[16+5+4\right]+(3)\left[32+9+4\right]+0+(-1)\left[8-4+1\right]$$

$$=-25+135-5$$

$$=105.$$