More on matrix inverses

B is the inverse of A if AB = I and BA = I. (Both statements must be true.) However, it can be shown that if AB = I for an $n \times n$ matrix A, then BA = I as well.

If $B=A^{-1}$, then B is the only inverse of A (inverse matrices are unique.) We can therefore call A^{-1} "the" inverse of matrix A, such that $A^{-1}A = I$ and $AA^{-1} = I$. (Know this proof - Theorem 1 on page 186 of the textbook.)

An $n \times n$ matrix A has an inverse A^{-1} if we can row-reduce [A | I] to [I | B], in which case $B=A^{-1}$.

In other words - if A can be transformed into I_n via elementary row operations (i.e. if A is row-equivalent to I_n), then A is invertible.

If A and B are invertible matrices with the same size, then
AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Note that $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$
$= AIA^{-1}$
$= AA^{-1} = I$.

And $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$
$= B^{-1}IB = B^{-1}B = I$.

This idea can be extended to any number of matrices:

$(ABCD)^{-1}=D^{-1}C^{-1}B^{-1}A^{-1}$ ... and so on.

$A^0 = I$ and $A^n = AA \ldots A$ (n factors, where n is a nonnegative integer)

Assuming A is invertible, $(A^n)^{-1} = A^{-1}A^{-1} \ldots A^{-1}$ (n factors),
so we say $(A^n)^{-1} = (A^{-1})^n = A^{-n}$.

$A^rA^s = A^{r+s} = A^{s+r} = A^sA^r$ and $(A^r)^s = A^{rs}$

$(kA)^{-1} = k^{-1}A^{-1}$ (k is a scalar and A is invertible)

$A = \begin{bmatrix} 7 & 3 \\ 9 & 4 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 4 & -3 \\ -9 & 7 \end{bmatrix}$

$2A = \begin{bmatrix} 14 & 6 \\ 18 & 8 \end{bmatrix}$ $(2A)^{-1} = \frac{1}{112-108}\begin{bmatrix} 8 & -6 \\ -18 & 14 \end{bmatrix}$

should be $\frac{1}{2}\begin{bmatrix} 4 & -3 \\ -9 & 7 \end{bmatrix}$

If the $n \times n$ matrix A is invertible, then for any n-vector $\vec{b}$ the system

$$A\vec{x} = \vec{b} \qquad (1)$$

(row-equivalent to I, so no free vars.)

has the unique solution

$$\vec{x} = A^{-1}\vec{b}$$

obtained by multiplying both sides of (1) by the inverse A^{-1}.

Also ... if the coefficient matrix A of the $n \times n$ homogeneous system $A\vec{x} = \vec{0}$ reduces such that there are no free variables, then

(1) The system has only the trivial solution $\vec{x} = \vec{0}$.

(2) The matrix A is row-equivalent to the $n \times n$ identity matrix.

(3) The matrix A is invertible.

PROPERTIES OF NON-SINGULAR MATRICES:

The following properties of an $n \times n$ matrix are equivalent:

1) A is invertible.
2) A is row equivalent to the $n \times n$ identity matrix I.
3) $Ax = b$ is consistent.
4) $Ax = b$ has a unique solution.
5) $Ax = 0$ has only the trivial solution.

Also recall the structure of matrix multiplication:

$$\underset{2\times 3}{\begin{array}{c} \\ r_1 \rightarrow \\ r_2 \rightarrow \end{array}\begin{bmatrix} 2 & 1 & 2 \\ 3 & 0 & -2 \end{bmatrix}} \; \underset{3\times 4}{\begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \\ \end{array} \atop \begin{bmatrix} 4 & 2 & 0 & 1 \\ 1 & 2 & -7 & 0 \\ 5 & 3 & -3 & 1 \end{bmatrix}} = \begin{bmatrix} \boxed{\begin{matrix} \vec{r}_1\cdot\vec{c}_1 \\ \vec{r}_2\cdot\vec{c}_1 \end{matrix}} & \begin{matrix} \vec{r}_1\cdot\vec{c}_2 \\ \vec{r}_2\cdot\vec{c}_2 \end{matrix} & \begin{matrix} \vec{r}_1\cdot\vec{c}_3 \\ \vec{r}_2\cdot\vec{c}_3 \end{matrix} & \begin{matrix} \vec{r}_1\cdot\vec{c}_4 \\ \vec{r}_2\cdot\vec{c}_4 \end{matrix} \end{bmatrix}$$

$$A \qquad\qquad B$$

$$= \begin{bmatrix} A\vec{c}_1 & A\vec{c}_2 & A\vec{c}_3 & A\vec{c}_4 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\quad \vec{r}_1 B \quad} \\ \boxed{\quad \vec{r}_2 B \quad} \end{bmatrix}$$

Some curiosities:

$(AB)^{-1} = B^{-1}A^{-1}$ (assuming A, B are invertible)

$(A+B)^{-1} = ??$ << No rule for this!

Let A, B be $n \times n$ matrices.

$(A+B)^2$

$(a+b)^2 = a^2 + 2ab + b^2$.

$(A+B)(A+B)$

$= AA + AB + BA + BB$

$= A^2 + AB + BA + B^2$

STOP.

$= A^2 + 2AB + B^2$ (only if $AB = BA$ ... which isn't always true!)