The "Three Possibilities" (pg. 167)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

"$m \times n$" system.

Upon reducing this system to RREF (reduced row-echelon form) via Gauss-Jordan Elimination, we always end up with the following form:

$$\underbrace{\begin{cases} x_{j1} \qquad\qquad + \sum c_{1j}x_j = d_1 \\ \quad x_{j2} \qquad + \sum c_{2j}x_j = d_2 \\ \qquad \ddots \qquad\qquad \vdots \\ \qquad\quad x_{jr} + \sum_{j=r+1}^{n} c_{rj}x_j = d_r \\ \qquad\qquad\qquad 0 = d_{r+1} \\ \qquad\qquad\qquad 0 = d_{r+2} \\ \qquad\qquad\qquad \vdots \\ \qquad\qquad\qquad 0 = d_m \end{cases}}_{m \text{ eqns.}}$$

leading variables

collection of all free variables

- If any of the $d_{r+1}, d_{r+2}, \ldots, d_m \neq 0$, we have no solution (i.e., the system is inconsistent).

Assuming $d_{r+1} = d_{r+2} = \cdots = d_m = 0$:

- If $r = n$, we have no free variables, and therefore we have a single unique solution
- If $r < n$, we have free variables and can paramaterize to obtain infinitely many solutions.

$r = \#$ of leading vars.
$n - r = \#$ of free vars.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 3 & 4 \end{array}\right]$$

$r<n$, free vars, infinite sols.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

free vars, $r<n$, no sols.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r<n$, free vars, infinite sols.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$r=n$
no free vars,
unique sol.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right]$$

$r<n$, free vars
infinite sols

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$r=n$, no free vars, unique sol.

## The "Three Possibilities" (pg. 167)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

"$m \times n$" system.

Upon reducing this system to RREF (reduced row-echelon form) via Gauss-Jordan Elimination, we always end up with the following form:

$$m \text{ eq'ns.} \left\{ \begin{array}{l} x_{j_1} \qquad\qquad + \sum c_{1j}x_j = d_1 \\ \qquad x_{j_2} \qquad + \sum c_{2j}x_j = d_2 \\ \qquad\qquad \ddots \qquad\qquad \vdots \\ \qquad\qquad x_{j_r} + \sum_{j=r+1}^{n} c_{rj}x_j = d_r \\ \qquad\qquad\qquad 0 = d_{r+1} \\ \qquad\qquad\qquad 0 = d_{r+2} \\ \qquad\qquad\qquad\quad \vdots \\ \qquad\qquad\qquad 0 = d_m \end{array} \right.$$

$\uparrow \; \uparrow \; \cdots \; \uparrow$ leading variables

$\uparrow$ collection of all free variables

If the $d_1 = d_2 = \cdots = d_m = 0$, we have

- At least the trivial solution
- Only the trivial solution if $r = n$.
- The trivial solution & infinitely many others if $r < n$.

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 1 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

free

## Systems

The matrix equation

$$A\vec{x} = \vec{b}$$

represents a system such as

$$\underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\underset{2\times 1}{\begin{bmatrix} (1x+2y) \\ (3x+7y) \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} \Rightarrow \begin{matrix} x+2y=2 \\ 3x+7y=3 \end{matrix}$$

(we've solved this via Gauss-Jordan elimination... RREF)

Previously, you probably used the <u>inverse</u> matrix $A^{-1}$.

If $A\vec{x} = \vec{b}$, $\qquad 3x = 6$

then $A^{-1}A\vec{x} = A^{-1}b \qquad \frac{1}{3}\cdot 3x = \frac{1}{3}\cdot 6$

$I\vec{x} = A^{-1}b \qquad 1x = \frac{1}{3}\cdot 6$

$\boxed{\vec{x} = A^{-1}b} \qquad x = \frac{1}{3}\cdot 6$

How do we find a matrix $B$ such that $BA = I_n$? (And in general, we want the inverse to work in both directions, i.e., we want

$$BA = I_n \text{ and } AB = I_n)$$

$$(m \times n)(n \times p) = (m \times p) \qquad \overbrace{(n \times p)(m \times n)}^{\text{requires } p=m} = n \times n$$

but since $I_n$ is $n \times n$, $m = p = n$,

so $A$ & $B$ have to be $n \times n$ as well.

So... how do we find $B$ such that

$$AB = I = BA\,?$$

Find the inverse of $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$.

$$\underset{2\times 2}{\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} a-3c & b-3d \\ -2a+6c & -2b+6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a - 3c = 1 \qquad b - 3d = 0$$
$$-2a + 6c = 0 \qquad -2b + 6d = 1$$

$$2a - 6c = 2$$
$$-2a + 6c = 0$$
$$0 = 2 \quad !!$$

$$0 = 1 \quad !!$$

* Some matrices do <u>not</u> have an inverse.

We call such matrices "non-invertible" or "singular."

Find an inverse for $A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

3x3, (each $\vec{b}_i$ is a 3x1 column)

One way to view matrix multiplication is as follows:

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} = [4\,3\,2]\vec{b}_1 & c_{12} = [4\;3\;2]\vec{b}_2 & c_{13} = [4\;3\;2]\vec{b}_3 \\ c_{21} = [5\;6\;3]\vec{b}_1 & c_{22} = [5\;6\;3]\vec{b}_2 & c_{23} = [5\;6\;3]\vec{b}_3 \\ c_{31} = [3\;5\;2]\vec{b}_1 & c_{32} = [3\;5\;2]\vec{b}_2 & c_{33} = [3\;5\;2]\vec{b}_3 \end{bmatrix}$$

$$= \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

So we have

$$\begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \vec{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \vec{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We would solve each of these systems via elimination.

Since the three coefficient matrices are all the same, we can solve all ~~three~~ systems by using the same steps!!

$$\left[\begin{array}{ccc|ccc} 4 & 3 & 2 & 1 & 0 & 0 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\text{rref}\left(\left[\underbrace{A}\ \middle|\ \underbrace{I}\right]\right) \longrightarrow \left[\, I \mid A^{-1} \right]$$

$$\left[\begin{array}{ccc|ccc} 4 & 3 & 2 & 1 & 0 & 0 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right] -R_3 \quad \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -5R_1 \\ -3R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 16 & 3 & -5 & 1 & 5 \\ 0 & 11 & 2 & -3 & 0 & 4 \end{array}\right] \begin{array}{l} \\ \cdot 11 \\ 16 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 176 & 33 & -55 & 11 & 55 \\ 0 & 176 & 32 & -48 & 0 & 64 \end{array}\right] \begin{array}{l} \\ \\ -R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 176 & 33 & -55 & 11 & 55 \\ 0 & 0 & -1 & 7 & -11 & 9 \end{array}\right] \begin{array}{l} \\ (\div 11) \\ \cdot(-1) \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 16 & 3 & -5 & 1 & 5 \\ 0 & 0 & 1 & -7 & 11 & -9 \end{array}\right] \begin{array}{l} \\ -3R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 16 & 0 & 16 & -32 & 32 \\ 0 & 0 & 1 & -7 & 11 & -9 \end{array}\right] \begin{array}{l} \\ \div 16 \\ \\ \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 & 11 & -9 \end{array}\right] \begin{array}{l} +2R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 3 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 & 11 & -9 \end{array}\right]$$

$$I \quad | \quad A^{-1}$$

We can check:

$$\begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To find the inverse of an $n \times n$ matrix $A$, reduce $[A \mid I_n]$ to $[I_n \mid B]$.

$B$ will be the inverse of $A$, i.e.,

$$AB = I = BA.$$

<u>Proof</u> that matrix inverses are unique.

If matrix $A$ is invertible, then there exists precisely one matrix $B$ such that $AB = BA = I$.

Common approach to proving uniqueness is to assume that there is some other matrix that also works.

<u>Proof</u>: Assume $C$ is some (possibly different) inverse of $A$. This means

$$AC = CA = I.$$

We also know $AB = BA = I$.

$$(BA)C = (I)C$$

"moment of proof"

matrix mult. is assoc.

$$B(AC) = (I)C$$

b/c $C$ is an inverse

$$BI = IC$$

$$B = C$$

$\therefore$ our assumption that $C$ is some (other?) inverse leads to the conclusion that $C$ is the same as $B$.

- pg. 106: Shortcut for 2x2 inverses.
- pg. 107: Theorem 3.
  - $(A^{-1})^{-1} = A$.
  - $(A^n)^{-1} = (A^{-1})^n$
  - $(AB)^{-1} = B^{-1}A^{-1}$ ← (know this proof)

## Solving Systems

If $A\vec{x} = \vec{b}$,

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

- limited to square matrices that have inverses!
- Slow!