Recall:

$$\begin{cases} x_1 + 3x_2 - 15x_3 + 7x_4 = 0 \\ x_1 + 4x_2 - 19x_3 + 10x_4 = 0 \\ 2x_1 + 5x_2 - 26x_3 + 11x_4 = 0 \end{cases}$$

We've done:

$$\left[\begin{array}{cccc|c} 1 & 3 & -15 & 7 & 0 \\ 1 & 4 & -19 & 10 & 0 \\ 2 & 5 & -26 & 11 & 0 \end{array}\right] \leftarrow \text{"augmented coefficient matrix"}$$

$$\begin{bmatrix} 1 & 3 & -15 & 7 \\ 1 & 4 & -19 & 10 \\ 2 & 5 & -26 & 11 \end{bmatrix} \leftarrow \text{"coefficient matrix"}$$

This reduces via Gauss-Jordan (rref) to:

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & -3 & -2 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

LEAD: ✓ ✓

FREE: ✓ ✓

Let $x_3 = s$, $x_4 = t$. (parameterize on the free variables)

$$x_2 = 4s - 3t$$

$$x_1 = 3s + 2t$$

So our solution is (x_1, x_2, x_3, x_4)

$$= (3s+2t,\ 4s-3t,\ s,\ t).$$

Recall:

$$\begin{cases} x_1 + 3x_2 - 15x_3 + 7x_4 = 0 \\ x_1 + 4x_2 - 19x_3 + 10x_4 = 0 \\ 2x_1 + 5x_2 - 26x_3 + 11x_4 = 0 \end{cases}$$

So our solution is (x_1, x_2, x_3, x_4)

$= (3s+2t,\ 4s-3t,\ s,\ t)$.

Matrix multiplication:

$\vec{a} = [a_1\ a_2\ a_3\ \cdots\ a_n]$ $\quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$

row vector $\quad$ $1 \times n$ matrix $\quad$ column vector $\quad$ $n \times 1$ matrix.

$$\vec{a}\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n$$

Ex: If $\vec{u} = [2\ 3\ 5]$ and $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$

Then $\vec{u}\vec{v} = (2)(-1) + (3)(0) + (5)(3) = 13$

In a similar way,

$$\begin{cases} x_1 + 3x_2 - 15x_3 + 7x_4 = 0 \\ x_1 + 4x_2 - 19x_3 + 10x_4 = 0 \\ 2x_1 + 5x_2 - 26x_3 + 11x_4 = 0 \end{cases}$$

$$\text{Row } 1 = \begin{bmatrix} 1 & 3 & -15 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\text{Row } 2 = \begin{bmatrix} 1 & 4 & -19 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\text{Row } 3 = \begin{bmatrix} 2 & 5 & -26 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

<u>Summarizing:</u>

$$\begin{bmatrix} 1 & 3 & -15 & 7 \\ 1 & 4 & -19 & 10 \\ 2 & 5 & -26 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3×4 $\quad$ 4×1 $\quad$ 3×1

must match in order to multiply.

$$a_i b_j = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Try one:

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 4 & 0 \\ 2 & 1 & 0 \\ 2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 21 & 6 \\ 10 & 9 & 0 \end{bmatrix}$$

2×3 $\qquad$ 3×3 $\qquad$ 2×3

Compatible? yes.

ab_{23} (use row 2 of A, col 3 of B)

$2 \cdot 0 + 1 \cdot 0 + 0 \cdot 3$

ab_{12} (row 1 of A, col 2 of B)

$3 \cdot 4 + 1 \cdot 1 + 2 \cdot 4 \leftarrow$

Recall:

$$\begin{cases} x_1 + 3x_2 - 15x_3 + 7x_4 = 0 \\ x_1 + 4x_2 - 19x_3 + 10x_4 = 0 \\ 2x_1 + 5x_2 - 26x_3 + 11x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -15 & 7 & 0 \\ 1 & 4 & -19 & 10 & 0 \\ 2 & 5 & -26 & 11 & 0 \end{array}\right] \leftarrow \text{Augmented Coefficient Matrix}$$

Summarizing:

$$\underbrace{\begin{bmatrix} 1 & 3 & -15 & 7 \\ 1 & 4 & -19 & 10 \\ 2 & 5 & -26 & 11 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{0}}$$

"Coefficient Matrix"

$$A\vec{x} = \vec{0} \longrightarrow \text{"Matrix Equation"}$$

Recall:

$$\begin{cases} x_1 + 3x_2 - 15x_3 + 7x_4 = 0 \\ x_1 + 4x_2 - 19x_3 + 10x_4 = 0 \\ 2x_1 + 5x_2 - 26x_3 + 11x_4 = 0 \end{cases}$$

Summarizing:

$$\underbrace{\begin{bmatrix} 1 & 3 & -15 & 7 \\ 1 & 4 & -19 & 10 \\ 2 & 5 & -26 & 11 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{0}}$$

"Coefficient Matrix"

$A\vec{x} = \vec{0} \longrightarrow$ "Matrix Equation"

So our solution is (x_1, x_2, x_3, x_4)
$= (3s+2t,\ 4s-3t,\ s,\ t)$.

The matrix equation $A\vec{x} = \vec{0}$ treats the solution as a column vector, i.e.,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s+2t \\ 4s-3t \\ s \\ t \end{bmatrix}$$

$$= s\begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

"Vector form" of the solution.

Note that any (s,t) pair will create another solution.

$$(s,t) = (1,1) \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

In the end, all solutions will be a "linear combination" of the vectors $(3,4,1,0)$ and $(2,-3,0,1)$, i.e., a sum of scalar multiples of these vectors.

"Rules of Matrix Algebra"

$A+B = B+A$ addition is commutative

$A+(B+C) = (A+B)+C$ addition is associative

$A(BC) = (AB)C$ multiplication is associative

(order still matters... ABC)

$A(B+C) = AB + AC$
$(A+B)C = AC + BC$
multiplication distributes over addition.

"Non-Rules"

- AB and BA are not necessarily equal.
  (mult. is not commutative)

$A \quad B$ exists, and AB is 2×4.
$(2\times3)\,(3\times4)$

But BA d.n.e.
$(3\times4)(2\times3)$
$4 \neq 2$

- Matrix multiplication is not commutative.
  (i.e. AB does not have to equal BA).

- Cancellation

  $AB = AC$ does not have to imply that $B = C$.

- Zero Product Property

  AB can equal the 0 matrix even if $A \neq 0$ and $B \neq 0$.

<u>Cancellation</u> (doesn't hold)

$$A = \begin{bmatrix} 4 & 1 & -2 & 7 \\ 3 & 1 & -1 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 \\ 3 & -1 \\ -2 & 4 \\ 2 & -3 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ -2 & 3 \\ 1 & -3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 25 & -10 \\ 18 & -5 \end{bmatrix} = AC \quad \text{even though } B \neq C.$$

<u>Zero Product Property</u> (doesn't hold)

$$\text{Let } D = B - C = \begin{bmatrix} -2 & 1 \\ 1 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$AD = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{even though } A \neq 0,\ D \neq 0.$$

2 special matrices:

Additive Identity (zero matrix)

$\mathbf{0}$ matrix is an $m \times n$ matrix full of zeros. (does not have to be square $n \times n$)

2×2 "zero matrix" is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Note that $A + 0 = A = 0 + A$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Multiplicative Identity

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

2×2 "identity matrix" - denoted I_2.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$ (must be $n \times n$.)

Note that $AI = A = IA$

Matrix - singular

Matrices - plural