"Linear Algebra" - Systems of Linear Equations and Matrices.

$$\begin{cases} 3x - 7y = 6 \\ 2x + y = 7 \end{cases} \longrightarrow \underbrace{\left[\begin{array}{cc|c} 3 & -7 & 6 \\ 2 & 1 & 7 \end{array}\right]}_{\text{"augmented coefficient matrix"}}$$

$$\begin{bmatrix} 3 & -7 \\ 2 & 1 \end{bmatrix} \longrightarrow \text{"coefficient matrix"}$$

A Matrix is an arranged array of elements with r rows and c columns.

The "size" or "shape" of the augmented coefficient matrix above is 2×3.
(rows x cols).

Matrix — singular
Matrices — plural

$$\begin{cases}3x-7y=6\\2x+y=7\end{cases} \qquad \left[\begin{array}{cc|c}3&-7&6\\2&1&7\end{array}\right]$$

We use "elementary row operations"

1. add a constant multiple of one row to another row
2. multiply a row by a nonzero constant
3. Swap rows

$$\begin{cases}3x-7y=6\\2x+y=7\end{cases} \xrightarrow[\text{eq}2\cdot 7]{} \begin{cases}3x-7y=6\\14x+7y=49\end{cases} \xrightarrow[\text{eq}1+\text{eq}2]{}$$

$$\begin{bmatrix}3&-7&6\\2&1&7\end{bmatrix} \xrightarrow{\text{(prop 2)}} \begin{bmatrix}3&-7&6\\14&7&49\end{bmatrix}$$

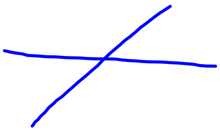
$$\begin{cases}17x+0y=55\\14x+7y=49\end{cases} \implies x=\frac{55}{17}.$$

$$\begin{bmatrix}17&0&55\\14&7&49\end{bmatrix}$$

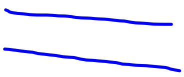
and $y = 7-2\left(\frac{55}{17}\right) = \frac{9}{17}$

$\therefore \left(\frac{55}{17}, \frac{9}{17}\right)$ is a solution.

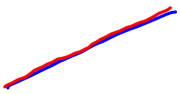
Possibilities (for linear systems):



⟵ two lines cross once...
(one soln)



⟵ do not cross
(no sols)



⟵ coincident lines
(infinitely many sols)

$$\begin{cases} 2x+6y=4 \\ 3x+9y=7 \end{cases} \qquad \left[\begin{array}{cc|c} 2 & 6 & 4 \\ 3 & 9 & 7 \end{array}\right] \xrightarrow{\cdot\frac{1}{2}} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & 9 & 7 \end{array}\right] -3R_1$$

$$\begin{array}{c} \begin{array}{ccc} x & y & = \end{array} \\ \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & 1 \end{array}\right] \end{array} \longleftarrow \text{not possible } (0x+0y=1).$$

algebraically: (no solution)

$\boxed{0 \cdot \text{variables} = \text{non-zero } \#}$

geometrically: parallel lines

(we say the system is "inconsistent")

$$\begin{cases} 2x+6y=4 \\ 3x+9y=6 \end{cases} \xrightarrow{eq_2 - \frac{3}{2}eq_1} \begin{cases} 2x+6y=4 \\ 0x+0y=0 \end{cases} \quad \left[\begin{array}{cc|c} 2 & 6 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} y=\frac{4-2x}{6} \\ y=\frac{6-3x}{9} \end{cases}$$

$$\boxed{0 \cdot \text{variables} = 0}$$

row of zeros in augmented coeff. matrix.

algebraically: <u>any</u> (all) ordered pairs would work (infinitely many solutions). (as long as they still satisfy the equation $2x+6y=4$).

geometrically: same (coincident) lines.

In cases where we have infinitely many sols, we often "parameterize" in terms of some other independent variable.

Since $2x+6y=4$, let $y=t$.

then $x=\frac{4-6t}{2}$

i.e. $(x,y)=(2-3t,t)$

leading variable ... "pivot"

$$\begin{cases} x + 2y + z = 4 \\ 3x + 8y + 7z = 20 & -3\text{eq}1 \\ 2x + 7y + 9z = 23 & -2\text{eq}1 \end{cases}$$

"Gaussian Elimination"

goal is to obtain triangular form:

$$\begin{array}{l} \_\_ + \_\_ + \_\_ = \_\_ \\ \qquad \_\_ + \_\_ = \_\_ \\ \qquad\qquad \_\_ = \_\_ \end{array}$$

or "echelon form"

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right] & \begin{array}{l} \\ -3R_1 \\ -2R_1 \end{array} \end{array}$$

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{array}\right] & \begin{array}{l} \\ \cdot \frac{1}{2} \\ \\ \end{array} \end{array}$$

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{array}\right] & \begin{array}{l} \\ \\ -3R_2 \end{array} \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

leading entries (pivots) are all 1 ... this can be helpful

We now see that $z = 3$. We "back-substitute" to get the rest ...

Row 2 indicates $y + 2z = 4$, so $y = 4 - 2(3)$
$= -2$.

Row 1 indicates $x + 2y + 1 = 4$, so
$x = 4 - 2(-2) - (3) = 5$.

$(x, y, z) = (5, -2, 3)$.

However, we can continue:

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} -R_3 \\ -2R_3 \\ \end{array} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{array}{l} -2R_2 \\ \\ \\ \end{array} \longrightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{array}{l} \Longrightarrow x = 5 \\ \Longrightarrow y = -2 \\ \Longrightarrow z = 3 \end{array}$$

This is called "Gauss-Jordan elimination".

ECHELON FORM

The above theorem is what allows us to solve a system of equations by using elementary row operations to get its coefficient matrix in a specific "triangular" form, called echelon form. An **echelon matrix** **E** is one having the following two properties (and is sometimes called a **row-echelon matrix**):

1. Every row of **E** that consists entirely of zeros (if any) lies beneath every row that contains a nonzero element.
2. In each row of **E** that contains a nonzero element, the first nonzero element (called the **leading entry**) lies strictly to the right of the first nonzero element in the preceding row (if there is a preceding row).

$$\begin{array}{ccccc|c} w & x & y & z & = & \# \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

"free variables" (no leading entry)

"leading variables" (leading entries)

(when solving systems, we parameterize on free variables).

If matrix B can be obtained from matrix A via a finite number of elementary row operations,

(1) A and B are "<u>row equivalent</u>".

(2) If A and B are augmented coefficient matrices of linear systems, the two systems have the same solution set.

TRY:

Reduce the augmented coefficient matrix of the system and use it to find the solution.

$$\begin{cases} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 10 \\ 2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 = 7 \\ 3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 = 27 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right] \begin{array}{l} \\ -2R_1 \\ -3R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 2 & -1 & 8 & -13 \\ 0 & 0 & 1 & 0 & 2 & -3 \end{array}\right] \begin{array}{l} \\ -2R_3 \\ \\ \end{array}$$

3×5 matrix

eqns   variables

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 0 & -1 & 4 & -7 \\ 0 & 0 & 1 & 0 & 2 & -3 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 & 4 & -7 \end{array}\right] \longleftarrow \text{"echelon form"}$$

$\therefore$ back substitute.

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

leading vars. (x_1, x_3, x_4)

free vars... (x_2, x_5) let these be parameters and solve for the leading vars in terms of these parameters.

Let $x_2 = r$, $x_5 = t$.

$-x_4 + 4x_5 = -7$

$\Rightarrow x_4 = 7 + 4x_5 = 7 + 4t$. (b/c $x_5 = t$).

$x_3 + 2x_5 = -3$

$\Rightarrow x_3 = -3 - 2t$.

$x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 10$

$\Rightarrow x_1 = 10 + 2r - 3x_3 - 2x_4 - t$

$= 10 + 2r - 3(-3-2t) - 2(7+4t) - t$

$= 5 + 2r - 3t$

Infinitely many solutions:

$(x_1, x_2, x_3, x_4, x_5) = (5+2r-3t,\ r,\ -3-2t,\ 7+4t,\ t)$.

Possible intersections of planes:

Image from http://geomalgorithms.com/a05-_intersect-1.html