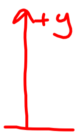
Equations of Motion

We've discussed motion along a line:

Assume constant acceleration $a(t) = -g$.



If $y(t)$ is an object's position, then the change in position per unit time is called velocity, and $v(t) = \frac{dy}{dt}$. Also, the change in velocity per unit time is called acceleration, and $a(t) = \frac{dv}{dt} = \frac{d^2y}{dt^2}$.

Under the assumption of constant acceleration $a(t) = -g$, and with $v(0) = v_0$ and $y(0) = y_0$, we have:

$$a(t) = -g$$

$$v(t) = \int a(t)\,dt = \int -g\,dt = -gt + C$$

Since $v(0) = v_0 = -g(0) + C \Rightarrow \boxed{C = v_0.}$

So $\boxed{v(t) = -gt + v_0}$

$$y(t) = \int v(t)\,dt = \int (-gt + v_0)\,dt = \frac{-gt^2}{2} + v_0 t + C_1$$

Since $y(0) = y_0 = \frac{-g(0)^2}{2} + v_0(0) + C_1 \Rightarrow \boxed{C_1 = y_0}$

So $\boxed{y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0.}$

Initial velocity

Initial position

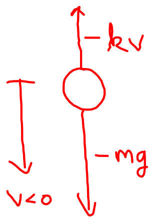
In mks: $g = 9.8\ m/s^2$

In fps:

$g = 32\ ft/s^2$

<u>What about air resistance?</u>

We generally treat air resistance as something that works <u>against</u> velocity.



$k, g > 0.$

$$ma = \sum F$$

$$m\frac{dv}{dt} = -kv - mg$$

$$\boxed{\frac{dv}{dt} = -\frac{k}{m}v - g}$$

Let $\alpha = \frac{k}{m}$.

So $\frac{dv}{dt} = -\alpha v - g$.

Let's solve via undetermined coefficients:

$$V_h = Ce^{-\alpha t}$$

Since the nonhomogeneous term is $-g$ (constant) we'll guess $v_p = A$. (So $v_p' = 0$).

So $0 = -\alpha A - g$

$$-\frac{g}{\alpha} = A,$$

and $v_p = -\frac{g}{\alpha}$, and $v(t) = v_p + v_h$

$$\boxed{v(t) = Ce^{-\alpha t} - \frac{g}{\alpha}.}$$

Also, $V(0) = v_0 \implies v_0 = C - \frac{g}{\alpha}$

$$\implies C = v_0 + \frac{g}{\alpha}.$$

$$\implies v(t) = \left(v_0 + \frac{g}{\alpha}\right)e^{-\alpha t} - \frac{g}{\alpha}$$

<u>Compare velocities:</u>

W/o resistance:

$v(t) = -gt + v_0$.

$\lim_{t\to\infty} v(t) = -\infty$

(no limit to velocity)

W/ resistance:

$v(t) = \left(v_0 + \frac{g}{\alpha}\right)e^{-\alpha t} - \frac{g}{\alpha}$

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left[\frac{v_0 + \frac{g}{\alpha}}{e^{\alpha t}} - \frac{g}{\alpha}\right]$$

$$= -\frac{g}{\alpha}$$

$$= \frac{-mg}{k} = V_\tau$$

("terminal velocity")

$$v(t) = \left(v_0 + \frac{g}{\alpha}\right)e^{-\alpha t} - \frac{g}{\alpha}$$

with $-\frac{g}{\alpha} = \frac{-mg}{k} = v_\tau$;

$$v(t) = (v_0 - v_\tau)e^{-\alpha t} + v_\tau .$$

Solve for $y(t)$.

$$y(t) = \int \left((v_0 - v_\tau)e^{-\alpha t} + v_\tau\right)dt$$

$$y(t) = \frac{(v_0 - v_\tau)e^{-\alpha t}}{-\alpha} + v_\tau t + C_1$$

with $y_0 = y(0)$:

$$y_0 = \frac{v_0 - v_\tau}{-\alpha} + C_1 \Rightarrow C_1 = y_0 + \frac{v_0 - v_\tau}{\alpha}$$

And finally, $y(t) = \frac{(v_0 - v_\tau)}{-\alpha}e^{-\alpha t} + v_\tau t + \left(y_0 + \frac{v_0 - v_\tau}{\alpha}\right)$

(compare to constant acceleration:

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0 \Big)$$

or $y(t) = \frac{v_0 + \frac{mg}{k}}{-\frac{k}{m}} e^{-\frac{k}{m}t} + \left(-\frac{mg}{k}\right)t + \left(y_0 + \frac{v_0 + \frac{mg}{k}}{\frac{k}{m}}\right)$

Actually, it's not just the velocity... we have resistance proportional to some power of the velocity:

$$\frac{dv}{dt} = -k v^{\theta} - g \, , \quad \text{typically, } 1 \leq \theta \leq 2$$

no longer linear in v :(.