Solve $\underbrace{\frac{dy}{dt} = -3y + 2e^{3t}}_{\text{nonhomogeneous}}$; $\quad y(0) = 2$.

① Associated homogeneous eqn: $\frac{dy}{dt} = -3y$

$\Rightarrow \boxed{y_h = Ce^{-3t}}$

② Choose y_p to "match" (sort of) the nonhomogeneous terms. Let $y_p = Ae^{3t}$.

Since we're assuming $y_p = Ae^{3t}$ is a soln, we should be able to substitute: note: $y_p' = 3Ae^{3t}$

$y_p' = -3y + 2e^{3t}$ will become

$3Ae^{3t} = -3(Ae^{3t}) + 2e^{3t}$

$6Ae^{3t} = 2e^{3t}$

$A = \frac{1}{3}$.

$\Rightarrow \boxed{y_p = \frac{1}{3}e^{3t}}$

"The" general soln is $y_p + ky_h$.

$\therefore\ y(t) = Ce^{-3t} + \frac{1}{3}e^{3t}$.

If $y(0) = 2$, $Ce^0 + \frac{1}{3}e^0 = 2 \Rightarrow C = 5/3$.

$$\boxed{y(t) = \frac{5}{3}e^{-3t} + \frac{1}{3}e^{3t}.}$$

$$\frac{dy}{dt} = -2y + \cos 3t + e^{t}$$

$$y_h = k e^{-2t}$$

Try $y_p = A\cos 3t + B\sin 3t + Ce^{t}$

$$y_p' = -3A\sin 3t + 3B\cos 3t + Ce^{t}$$

$$\underbrace{-3A\sin 3t + 3B\cos 3t + Ce^{t}}_{dy/dt} = \underbrace{-2\left(A\cos 3t + B\sin 3t + Ce^{t}\right)}_{-2y} + \cos 3t + e^{t}$$

$\sin 3t: \quad -3A = -2B \qquad B = \frac{3}{2}A$

$\cos 3t: \quad 3B = -2A + 1 \qquad \frac{9}{2}A = -2A + 1 \qquad A = \frac{2}{13}$

$e^{t}: \quad C = -2C + 1 \Rightarrow C = \frac{1}{3}. \qquad B = \frac{3}{13}$

$$y_p = \frac{2}{13}\cos 3t + \frac{3}{13}\sin 3t + \frac{1}{3}e^{t}$$

So $y = y_p + k y_h$

$$\boxed{y = \frac{2}{13}\cos 3t + \frac{3}{13}\sin 3t + \frac{1}{3}e^{t} + ke^{-2t}}$$

$$\frac{dy}{dt} = -2y + 3e^{-2t}$$

$y_h = ke^{-2t} \longrightarrow$ solves $\frac{dy}{dt} = -2y$.

$y_p = Ae^{-2t} \longrightarrow$ "duplicates" y_h

$\boxed{\text{try } y_p = \underbrace{At}_{u}\underbrace{e^{-2t}}_{v}.}$

$$\cancel{y_p' = -2Ae^{-2t}}$$

$$\cancel{-2Ae^{-2t} = -2Ae^{-2t} + 3e^{-2t}}$$

$$\cancel{0 = 3e^{-2t}} \leftarrow \text{no } A \text{ makes this true!}$$

$$y_p' = Ae^{-2t} - 2Ate^{-2t}$$

$y_p' = -2y + 3e^{-2t}$ becomes

$$\underbrace{Ae^{-2t} - \cancel{2Ate^{-2t}}}_{y_p'} = \underbrace{-\cancel{2Ate^{-2t}}}_{-2y_p} + 3e^{-2t}$$

$$\boxed{A = 3.}$$

$$\therefore \quad y(t) = ky_h + y_p$$

$$\boxed{y(t) = ke^{-2t} + 3te^{-2t}}.$$

"perturbation"

Check:

$$y'(t) = -2ke^{-2t} + 3e^{-2t} - 6te^{-2t}$$

$$= -2(ke^{-2t} + 3te^{-2t}) + 3e^{-2t}$$

$$= -2 \cdot y + 3e^{-2t}$$

Principles of linearity for homogeneous equations:

Suppose $y_1(t)$ and $y_2(t)$ both solve the homogeneous equation $y' + p(t)y = 0$.

- Then any constant multiple of y_1 or y_2 would also solve the equation.
- The sum $y_1 + y_2$ solves the equation.

<u>Proof</u>:

Assume y_1 & y_2 solve $y' + p(t)y = 0$.

So... $y_1' + p(t)y_1 = 0$ <u>And</u> $y_2' + p(t)y_2 = 0$

$\Rightarrow y_1' = -p(t)y_1$. $\Rightarrow y_2' = -p(t)y_2$.

Note that $(y_1 + y_2)' = y_1' + y_2'$

used hypotheses that y_1 & y_2 are sols.

$$\boxed{= (-p(t)y_1) + (-p(t)y_2)}$$

$$= -p(t)(y_1 + y_2)$$

And $(y_1 + y_2)' + p(t)(y_1 + y_2) = 0$.

$\therefore$ $y_1 + y_2$ is a soln to the homogeneous eqn. (This is called the "superposition principle")

Let $y = ky_1$. Then $y' = (ky_1)'$

$$= ky_1'$$

used the fact that y_1 is a soln.

$$\boxed{= k(-p(t)y_1)}$$

$$= -p(t)(ky_1)$$

So $(ky_1)' + p(t)(ky_1) = 0$

$\square' + p(t)\square = 0 \Rightarrow \square$ is a soln.

Ultimately, if $y_1, y_2, \ldots, y_n$ are all solutions to a lin. homog. eqn, then so is any solution of the form

$$y = a_1 y_1 + a_2 y_2 + \cdots + a_n y_n.$$

(a_i is constant).

Linearity Principles for nonhomogeneous eqns.

Given a nonhomogeneous equation (linear)

$$y' + p(t)y = f(t) \quad (1)$$

we call

$$y' + p(t)y = 0 \quad (2)$$

its "associated homogeneous equation".

If y_p solves (1) and y_h solves (2),

then $y_p + y_h$ solves (1) as well.

p for "particular"

h for "homogeneous"
(or sometimes y_c for "complimentary")

Suppose y_p solves $y' + p(t)y = f(t)$,

$(\text{so } y_p' = -p(t)y_p + f(t))$

And y_h Solves $y' + p(t)y = 0$

$(\text{So } y_h' = -p(t)y_h)$.

Then $y_p + y_h$ also solves $y' + p(t)y = f(t)$.

Why?

Let $y = y_p + y_h$. Then

$$y' = (y_p + y_h)' = y_p' + y_h'$$

used the hypotheses.

$$= \boxed{-p(t)y_p + f(t) + -p(t)y_h}$$

$$\Longrightarrow (y_p + y_h)' = -p(t)(y_p + y_h) + f(t)$$

$$\Longrightarrow (y_p + y_h)' + p(t)(y_p + y_h) = f(t)$$

$\therefore$ $y_p + y_h$ is a sol'n to the nonhomogeneous equation!

Finally, every solution to the linear nonhomogeneous equation is of the form $y = y_h + y_p$.

Proof:

Suppose y_p and y_g are both solutions to the nonhomogeneous equation $y' + p(t)y = f(t)$, and y_h is a solution to the associated homogeneous equation $y' + p(t)y = 0$.

Note that $\underline{\underline{y_g}} = y_p + (y_g - y_p)$

this represents any solution to the nonhomogeneous eqn.

Note that

$$(y_g - y_p)' + p(t)(y_g - y_p)$$
$$= y_g' - y_p' + p(t)y_g - p(t)y_p$$
$$= \underbrace{y_g' + p(t)y_g}_{} - \underbrace{(y_p' + p(t)y_p)}_{}$$
$$= f(t) \quad - \quad f(t)$$
$$= 0$$

$\therefore$ $y_g - y_p$ is simply a solution to the homogeneous equation!

RECAP:

- Any particular solution y_p can be written as $y_p = y_g + (y_p - y_g)$.
- But $y_p - y_g$ is just another y_h. (i.e., a homogeneous solution).
- So... we conclude that any particular solution y_p is a sum of another particular solution y_g and a homogeneous solution y_h... $y_p = y_g + y_h$.