Equations of Motion

Assume we have constant acceleration.

If $y(t)$ is position at time t, then

say m or ft

$y'(t)$ is rate of change in position.

(i.e., $y'(t) = v(t)$)

m/sec or ft/sec.

$y''(t) = v'(t)$ is rate of change in velocity

(i.e., $y''(t) = v'(t) = a(t)$)

$\frac{m/sec}{sec}$ $\frac{ft/sec}{sec}$

Assume we have constant acceleration.

So $a(t) = a$.

We can work backwards to obtain $v(t)$ and $y(t)$.

Since $v'(t) = a(t)$,

$$v(t) = \int a(t)\,dt = \int a\,dt = at + C$$

Now, if $v(0) = V_0$, we have

$$V_0 = a(0) + C \Longrightarrow C = V_0.$$

and $\boxed{v(t) = at + V_0}$

Then, since $y'(t) = v(t)$, we have

$$y(t) = \int v(t)\,dt = \int (at + v_0)\,dt$$

$$= \frac{at^2}{2} + v_0 t + C_1$$

and with $y(0) = y_0$, we have

$$y_0 = \frac{a(0)^2}{2} + v_0(0) + C_1 \implies C_1 = y_0 .$$

So $\boxed{y(t) = \frac{1}{2}at^2 + v_0 t + y_0}$

initial velocity

initial position

What about friction (e.g. air resistance?)

One common approach is to assume air resistance is proportional to velocity.

$\uparrow +y$

$-kv$ $\quad k>0$

$-mg$

$$\sum F = m \cdot a.$$

$$-kv - mg = m \cdot a$$

$$-kv - mg = m \cdot v'$$

$$\frac{dv}{dt} + \frac{k}{m} v(t) = -g$$

Call $\frac{k}{m} = \alpha$. Then

$$\frac{dv}{dt} = -g - \alpha v.$$

$$\int \frac{dv}{g+\alpha v} = -\int dt$$

$$\frac{1}{\alpha} \cdot \ln|g+\alpha v| = -t + C$$

$$\ln|g+\alpha v| = -\alpha t + C_1$$

$$|g+\alpha v| = e^{-\alpha t} e^{C_1}$$

$$g+\alpha v = \boxed{\pm e^{C_1}} e^{-\alpha t} \qquad (\text{with } v(0) = v_0)$$

$$\boxed{g+\alpha v_0 = \pm e^{C_1}}$$

$$g+\alpha v = \boxed{(g+\alpha v_0)} e^{-\alpha t}$$

$$v(t) = \left[(g+\alpha v_0)e^{-\alpha t} - g\right]\cdot\frac{1}{\alpha}$$

$$v(t) = \left[\left(\frac{g}{\alpha} + v_0\right)e^{-\alpha t} - \frac{g}{\alpha}\right]$$

$$\boxed{v(t) = \left[\left(\frac{mg}{k} + v_0\right)e^{-\frac{k}{m}t} - \frac{mg}{k}\right]}$$

In truth, the approach is to assume air resistance is proportional to v^θ where θ is typically $1 < \theta < 2$.

Constant acceleration:

$$v(t) = at + v_0$$

$$\lim_{t\to\infty} v(t) = \begin{matrix} \infty & (a>0) \\ -\infty & (a<0) \end{matrix}$$

But here, as $t \to \infty$,

$$v(t) \to -\frac{mg}{k} = V_\tau$$ "terminal velocity"

$$v(t) = \left[\left(\frac{mg}{k} + v_0\right)e^{-\frac{k}{m}t} - \frac{mg}{k}\right]$$

$v(t) = at + v_0$

What is $y(t)$?

For comparison w/ constant $a(t)$:
$y(t) = \frac{1}{2}at^2 + v_0 t + y_0$.

$$v(t) = (v_0 - v_\tau)e^{-\alpha t} + v_\tau$$

$$y(t) = \int v(t)\,dt = \frac{(v_0 - v_\tau)e^{-\alpha t}}{-\alpha} + v_\tau t + C_0$$

With $y(0) = y_0$, $\frac{(v_0 - v_\tau)e^0}{-\alpha} + v_\tau(0) + C_0 = y_0$

$$\boxed{C_0 = y_0 + \frac{v_0 - v_\tau}{\alpha}}$$

$$y(t) = \frac{(v_0 - v_\tau)}{-\alpha}e^{-\alpha t} + v_\tau t + \left(y_0 + \frac{v_0 - v_\tau}{\alpha}\right)$$

$$\boxed{y(t) = \frac{v_0 - v_\tau}{\alpha}\left(1 - e^{-\alpha t}\right) + v_\tau t + y_0}$$

$m = 10{,}000$
$g = 9.8\ \text{m/s}^2$
$k = 0.5$

$y_0 = 25{,}000$
$v_0 = 0$