1st-order equations (linear)

- integrating factor
- undetermined coefficients

Undetermined Coefficients:

- Find y_h (usually involves an arbitrary constant)

* Recognize $\frac{dy}{dt} + ky = 0 \Rightarrow y = Ce^{-kt}$

- Find y_p
- Create $y = y_h + y_p$
- Use initial condition to determine value of the arbitrary constant from y_h.

① $\frac{dy}{dt} = y + 3e^{-2t}$

Associated homogeneous eqn:

$\frac{dy}{dt} = y \rightarrow y_h = Ce^{t}$.

② $\frac{dy}{dt} = -4y + 3e^{-t}$

$\frac{dy}{dt} = -4y \rightarrow y_h = Ce^{-4t}$.

③ $\frac{dy}{dt} = y + \cos 2t$

$y_h = Ce^{t}$

④ $\frac{dy}{dt} = 2y + \sin 2t$

$y_h = Ce^{2t}$

⑤ $\frac{dy}{dt} = 4y - 5e^{4t}$

$y_h = Ce^{4t}$

⑥ $\frac{dy}{dt} = \frac{y}{2} + 4e^{t/2}$

$y_h = Ce^{t/2}$

② $\frac{dy}{dt} = -4y + 3e^{-t}$

- $\frac{dy}{dt} = -4y \longrightarrow y_h = Ce^{-4t}.$

- y_p — guess $y_p = Ae^{-t}.$

  $y' = -4y + 3e^{-t}$ becomes

  $-Ae^{-t} = -4Ae^{-t} + 3e^{-t}$

  $3Ae^{-t} = 3e^{-t}$

  $A = 1 \Longrightarrow \boxed{y_p = Ae^{-t} = e^{-t}.}$

- $y = Ce^{-4t} + e^{-t}.$

5) $\frac{dy}{dt} = 4y - 5e^{4t}$

- $y_h = Ce^{4t}$
- $y_p = Ae^{4t}$

note... in this case, our y_p is said to "duplicate" some or all of y_h...

So $y' - 4y = -5e^{4t}$ becomes

$$4Ae^{4t} - 4Ae^{4t} = -5e^{4t}$$

$$0 = -5e^{4t} \quad \ldots$$

- If y_p duplicates y_h, try multiplying by t... try $y_p = Ate^{4t}$

this creates extra terms via the product rule that hopefully allow us to solve for A.

So $y' - 4y = -5e^{4t}$ becomes

$$\left(Ate^{4t}\right)' - 4\left(Ate^{4t}\right) = -5e^{4t}$$

(product rule)

$$Ae^{4t} + \cancel{4At}e^{4t} - \cancel{4}Ate^{4t} = -5e^{4t}$$

$$Ae^{4t} = -5e^{4t}$$

$$A = -5. \longrightarrow \boxed{y_p = -5te^{4t}.}$$

$$\frac{dy}{dt} = t^2 + e^{4t} + \sin 2t + 2y$$

$$y_h = Ce^{2t}$$

Guess for y_p: $At^2 + Bt + C + D\sin 2t + E\cos 2t + Fe^{4t}$

Then $y_p' = 2At + B + 2D\cos 2t - 2E\sin 2t + 4Fe^{4t}$

And $y' - 2y$ should equal $t^2 + e^{4t} + \sin 2t$.

$$y' - 2y = 2At + B - 2(At^2 + Bt + C)$$
$$+ 2D\cos 2t - 2E\sin 2t - 2(D\sin 2t + E\cos 2t)$$
$$+ 4Fe^{4t} - 2Fe^{4t}$$
$$= t^2 + e^{4t} + \sin 2t$$

Constant: $B - 2C = 0$ $\quad C = -1/4$

t : $2A - 2B = 0$ $\quad B = -1/2$

t^2 : $-2A = 1 \longrightarrow A = -1/2$

$\cos 2t$: $2D - 2E = 0 \longrightarrow -4E = 1 \longrightarrow E = -1/4$

$\sin 2t$: $-2E - 2D = 1$ $\quad D = -1/4$

e^{4t} : $2Fe^{4t} = e^{4t} \longrightarrow F = 1/2$.

$$\boxed{\therefore\ y_p = -\frac{1}{2}t^2 - \frac{1}{2}t - \frac{1}{4} - \frac{1}{4}\sin 2t - \frac{1}{4}\cos 2t + \frac{1}{2}e^{4t}}$$

Mixing Revisited

Ex 5 page 53:

I.C.:

90 lbs of salt in 90 gals of H_2O.

- $2\ \text{lb}/\text{gal}$ in at $4\ \text{gal}/\text{min}$,
- out at $3\ \text{gal}/\text{min}$

How much salt is there when the tank becomes full? Translation: If $x(t)$ = amt. (lbs) of salt in the tank at time t, what is $x(30)$?

$$\frac{dx}{dt}\ \frac{\text{lbs}}{\text{min}} = \frac{2\text{lbs}}{\text{gal}} \cdot \frac{4\text{gal}}{\text{min}} - \frac{3\text{gal}}{\text{min}} \cdot \frac{x\ \text{lbs}}{90+t\ \text{gal}}$$

$$\frac{dx}{dt} = 8 - \left(\frac{3}{90+t}\right) \cdot x \qquad x(0) = 90\ \text{lbs.}$$

I'd use an integrating factor to solve...